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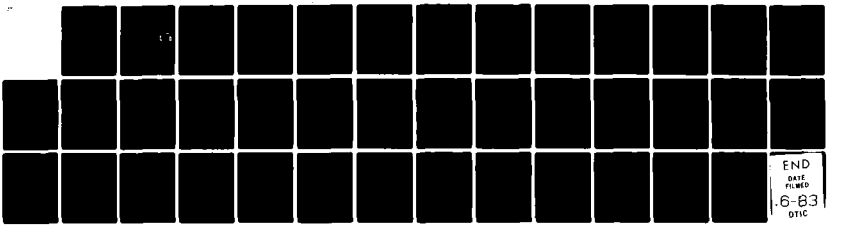

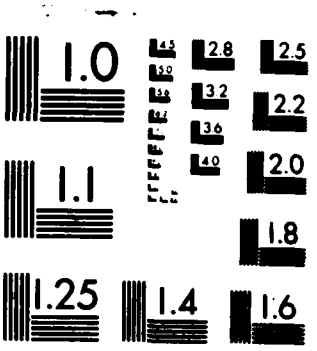

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Proving Precedence Properties: The Temporal Way

by

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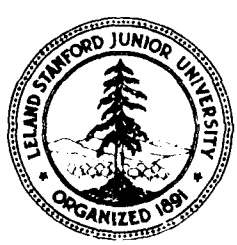

PROVING PRECEDENCE PROPERTIES: THE TEMPORAL WAY

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Abstract:

The paper explores the three important classes of temporal properties of concurrent programs: invariance, liveness and precedence. It presents the first methodological approach to the precedence properties, while providing a review of the invariance and liveness properties. The approach is based on the *unless* operator $\mathfrak{U}$, which is a weak version of the *until* operator $\mathcal{U}$. For each class of properties, we present a single complete proof principle. Finally, we show that the properties of each class are decidable over finite state programs.

1. INTRODUCTION

In studying temporal properties of programs, i.e., properties that go beyond partial correctness, an obvious hierarchy of such properties can be developed. One way of classifying the different sets in this hierarchy is by the syntax of the temporal formulas expressing them.

The first set in this hierarchy is the class of *invariance* properties (*safety* in the terminology of [L1]). These are the properties that can be expressed in terms of a formula of the form:

$$\Box\psi \quad \text{or} \quad \varphi \supset \Box\psi.$$

A formula of the first form, stated for a program P, says that every computation of P continuously satisfies ψ. In the case of the second form, the formula says that whenever φ is true, ψ is immediately realized and will hold continuously throughout the rest of the computation. Properties

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falling into this class include partial correctness, clean behavior (*error freedom*), mutual exclusion, and deadlock absence.

The second set in the hierarchy of properties is the class of *liveness* properties (*eventualities* in the terminology of [MP1]). These are properties that are expressible by temporal formulas of the form:

$$\Diamond \psi \quad \text{or} \quad \varphi \supset \Diamond \psi.$$

In both forms these formulas guarantee the occurrence of some event ψ, in the first case unconditionally and in the second case conditionally on an earlier occurrence of the event φ. Among the properties falling into this class are: total correctness, termination, accessibility, lack of individual starvation, and responsiveness.

While most of the researchers in the field tend to agree that these two classes are the first two rungs in a natural hierarchy, there is less of a consensus about what should be the next step in the hierarchy. In previous work we have proposed that the next class to be studied is that of *precedence* properties. In a broad sense, precedence properties are all the properties that are expressible using the *until* operator $\mathcal{U}$. To remind the reader, the expression $p\mathcal{U}q$, read "p until q", means that eventually q must happen and between now and then p must continuously hold.

A more mathematical formulation of this definition is given by:

Let $\sigma = s_0, s_1, s_2, \ldots$ be a sequence of states, then $p\mathcal{U}q$ is true for σ if there exists a $j \geq 0$ such that:

q is true for the sequence $s_j,\ s_{j+1},\ s_{j+2},\ \ldots$

(if q is a state property then q holds at s_j), and for every i, $0 \leq i < j$:

p is true for the sequence $s_i,\ s_{i+1},\ s_{i+2},\ \ldots$

(if p is a state property then p holds at s_i). Here, a state property is a property that depends only on the state and not on the full sequence. Note that in the special case that $j = 0$, then q is true on σ and no requirements for p are implied.

A derived operator is the *precede* operator $\mathcal{P}$ that can be defined by:

$$p\mathcal{P}q \equiv \sim((\sim p)\mathcal{U}q).$$

The meaning of this operator is that "p precede q", i.e., if q ever happens it cannot happen unless p occurs first (strictly before q). In contrast to $p\mathcal{U}q$ which requires that q eventually happens, $p\mathcal{P}q$ is automatically satisfied if q never happens.

We often use nested *until* expressions of the form

$$p_1\,\mathcal{U}\,(p_2\,\mathcal{U}\,(p_3\,\mathcal{U}\ldots(p_k\,\mathcal{U}q)\ldots)),$$

where $p_1, \ldots, p_k, q$ are state properties, i.e., formulas dependent only on the state and containing no temporal operators. By careful examination of the semantic definition of the until operator

we arrive at the interpretation that, stated at t_0, this expression means that there exist instants $t_1, \ldots, t_k$,

$$t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k,$$

such that:

p_1 holds in every t, $t_0 \leq t < t_1$

p_2 holds in every t, $t_1 \leq t < t_2$

$\vdots$

p_k holds in every t, $t_{k-1} \leq t < t_k$, and

q holds in t_k.

Thus, this expression predicts a period of continuous p_1 followed by a period of continuous p_2, and so on, until a period of continuous p_k, followed by an occurrence of q. Note that any of these periods may be empty by having $t_i = t_{i+1}$ for an empty $(i+1)$st period.

Since we are interested only in nested *until* expressions where the nesting is in the second argument, we can omit the parentheses and represent the expression above by:

$$p_1 \mathcal{U} p_2 \mathcal{U} p_3 \ldots p_k \mathcal{U} q.$$

The class of precedence properties that we consider are therefore formulas of one of the forms:

$p \supset (q \mathcal{P} r)$ — a *precede* formula

$p \supset (p_1 \mathcal{U} p_2 \mathcal{U} \ldots p_k \mathcal{U} q)$ — an *until* formula.

Several interesting properties fall into the broad class of precedence properties.

Example:

Let us consider a program G (granter) serving as an allocator of a single resource between several processes (requesters) $R_1, \ldots, R_k$ competing for the resource. Let each R_i communicate with G by means of two boolean variables: r_i and g_i. The variable r_i is set to *true* by the requester R_i to signal a request for the resource. Once R_i has the resource it signals its release by setting r_i to *false*. The allocator G signals R_i that the resource is granted to him by setting g_i to *true*. Having obtained a release signal from R_i, which is indicated by $r_i =$ *false*, some time later, it will reappropriate the resource by setting g_i to *false*.

Several obvious and important properties of this system belong to the invariance and liveness classes. For instance, the property

$$\Box\big((\Sigma_{i=1}^{k} g_i) \leq 1\big),$$

ensuring that the resource is granted to at most one requester at a time, is an invariant property. In summing boolean variables we treat *true* as 1 and *false* as 0. Similarly, the important property

$$r_i \supset \Diamond g_i,$$

which ensures *responsiveness*, is a liveness property. It guarantees that every request r_i will eventually be granted by setting g_i to *true*.

Let us, however, consider some precedence properties which are relevant to the specification of such a system.

(*a*) ***Absence of Unsolicited Response.***

An important but often overlooked desired feature is that the resource will not be granted to a party who has not requested it. (A similar property in the context of a communication network is that every message received must have been sent by somebody.) This is expressible by the temporal formula

$$\sim g_i \supset (r_i \mathsf{P} g_i).$$

The formula states that if presently g_i is false, i.e., R_i does not presently have the resource, then before the resource will be granted to R_i the next time, R_i must signal a request by setting r_i to *true*.

(*b*) ***Strict (FIFO) Responsiveness.***

Sometimes the weak commitment of eventually responding to a request is not sufficient. At the other extreme we may insist that responses are ordered in a sequence parallelling the order of arrival of the corresponding requests. Thus if requester R_i succeeded in placing his request before requester R_j the grant to R_i should precede the grant to R_j. A straightforward translation of this sentence yields the following intuitive but slightly imprecise expression:

$$(r_i \mathsf{P} r_j) \supset (g_i \mathsf{P} g_j).$$

A more precise expression which also better conforms to the general form of the class of properties we discuss in this paper is:

$$(r_i \wedge \sim r_j \wedge \sim g_j) \supset (\sim g_j \mathsf{U} g_i).$$

It states that if we ever find ourselves in a situation where r_i is presently on, and r_j and g_j are both off, then we are guaranteed to eventually get a g_i, and until that moment, no grant will be made to R_j. Note that $r_i \wedge \sim r_j$ implies that R_i's request preceded R_j's request, which has not materialized yet. We implicitly rely here on the assumption that once a request has been made it is not withdrawn until the request has been honored.

This assumption can also be made explicit as part of the specification, using another precedence expression:

$$r_i \supset g_i \mathsf{P} (\sim r_i).$$

Note that while all the earlier properties are requirements from the granter, and should be viewed as the "post-condition" part of the specification, this requirement is the responsibility of the requesters. It can be viewed as part of the "pre-condition" of the specification. Without this assumption, we could not hope to implement the granter in any reasonable way, since it would have to respond to very short and intermittent requests.

(c) Bounded Overtaking.

The requirement of *FIFO* responsiveness may sometimes be too restrictive and difficult to implement. Any program for the allocator that scans the requests in a certain polling order, $r_1, \ldots, r_k$ and then back to r_1 may respond to requests in, say, the order of their *detection* by the program. This order may be different from the arrival order. A more realistic requirement would allow deviations from the *FIFO* discipline, provided they are bounded. For example 1-bounded overtaking would say that for every i and j such that r_i preceded r_j, we may allow g_j to precede g_i at most *once*. *FIFO* responsiveness may then be regarded as 0-bounded overtaking. In order to express k-bounded overtaking we have to use nested until expressions.

The 1-overtaking property can be expressed by a nested *until* expression:

$$(r_i \wedge \sim r_j) \supset (\sim g_j) \mathcal{U} g_j \mathcal{U} (\sim g_j) \mathcal{U} g_i.$$

This expression predicts a period in which R_j does not have the resource, followed by a continuous period in which R_j has got the resource, followed by a period in which R_j does not have the resource, followed by a grant of the resource to R_i. Since any of these periods may be empty, the formula actually states that in the *worst case*, R_j may gain the resource at most *once* before R_i. ∎

Proofs of invariance properties for concurrent programs, have been extensively discussed in the literature (e.g., [OG], [K],[L1], [MP2]). Fewer suggestions have been made for approaches to proving liveness properties (e.g., [OL], [MP2], [MP3]).

In this work we address the problem of verifying properties of the precedence class. Our main conclusion is that the verification of precedence properties does not call for radically new ideas and can actually be viewed as a generalization of the approaches suggested for invariance and liveness properties. In fact, *precede* formulas are in many respects generalization of invariance properties, whereas *until* formulas can be established by a generalization of the proof principles for liveness properties.

To provide a proper framework, we first introduce an abstract operational model of concurrent programs. We then outline a proof system based on temporal logic; the system has been shown in [MP5] to be relatively complete for proving all properties of concurrent programs. We then discuss some derived proof principles that are tailored directly for the verification of precedence properties. The utility of these principles is demonstrated by proving several examples.

2. A COMPUTATIONAL MODEL

We start by defining an abstract computational model; the temporal logic properties will be stated and proven for computations over this model.

The abstract model consists of the following elements:

Σ — A *set of computation states.* This is a possibly infinite set. Every element $s \in \Sigma$ represents the full configuration of the computing system; for concrete programs each state includes the values of all the program variables as well as the program pointers for all the processes.

θ — The *initiality predicate*. We will only consider computations originating in a state s_0 such that $\theta(s_0)$ holds.

T — A finite set of *transitions*. With each transition $\tau \in T$ we associate a partial function $f_\tau : S \to 2^S$, where $f_\tau(s)$ yields all the possible outcomes of the transition τ on the state $s \in S$. A transition $\tau \in T$ is said to be *enabled* on a state s if $f_\tau(s) \neq \phi$; otherwise it is called *disabled* on s. A state s such that no transition $\tau \in T$ is enabled on it is called *terminal*.

J — The *justice family*. This is a (possibly empty) family of sets of transitions $J = \{T_1^J, \ldots, T_k^J\}$. Each set in J, $T_i^J \subset T$, is called a *justice set* and a justice requirement defined below is to be applid to the set T_i^J.

$\mathcal{F}$ — The *fairness family*. This is a (possibly empty) family of sets of transitions $\mathcal{F} = \{T_1^F, \ldots, T_\ell^F\}$. Each set in $\mathcal{F}$, $T_j^F \subset T$, is called a *fairness set* and a fairness requirement is to be applied to T_j^F.

An *initialized computation* of such a system is a sequence of states with labelled transitions:

$$\sigma: \quad s_0 \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} s_2 \xrightarrow{\tau_3} \ldots \quad \text{where } \tau_i \in T,$$

which satisfies the following requirements:

- *Maximality*. The sequence σ is maximal, i.e., either it is infinite or the last state s_k is terminal.
- *Initiality*. The first state s_0 satisfies the initiality predicate, i.e., $\theta(s_0) = \mathit{true}$.
- *State-to-State transition*. For each step $s_i \xrightarrow{\tau_{i+1}} s_{i+1}$ in σ we have that $s_{i+1} \in f_{\tau_{i+1}}(s_i)$.
- *Justice*. For each $T^J \in J$ we impose a *justice requirement*:
  - σ is finite, or
  - σ is infinite and contains an infinite number of states on which no transition in T^J is enabled, or
  - an infinite number of σ-steps are labelled by transitions in T^J.

  This corresponds to the notion that if for all states from a certain point on, some transition in T^J (not necessarily always the same) is always enabled, then some transition of T^J will be taken infinitely many times.
- *Fairness*. For each $T^F \in \mathcal{F}$ we impose a *fairness requirement*:
  - σ is finite, or
  - σ is infinite and from a certain point on no transition of T^F is enabled, or
  - some transition of T^F is taken infinitely many times.

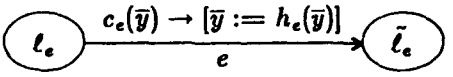
  This corresponds to the notion that if some transitions from T^F are enabled infinitely many times then some transitions from T^F are activated infinitely many times.

An ***admissible computation*** is any suffix of an initialized computation.

When considering a concrete computational system, we have to identify the five elements described above with more concrete objects. Since our example is based on a shared-variables computational model, we proceed with such identification for the ***shared-variables system***. Such a system has the form:

$$\overline{y} := g(\overline{x});\ [P_1 \parallel \ldots \parallel P_m],$$

where $\overline{y} = (y_1, \ldots, y_n)$ are the program (shared) variables, $\overline{x} = (x_1, \ldots, x_\ell)$ are the input variables, and $P_1, \ldots, P_m$ are the concurrent processes of the program. Each P_i is represented by a transition graph with nodes (locations) $L_i = (\ell_0^i, \ldots, \ell_t^i)$ and directed edges $E_i = \{e_1^i, \ldots, e_r^i\}$. The locations ℓ_0^i are the *entry* locations of P_i, respectively. Each edge $e \in E_i$ is labelled by an instruction:



whose meaning is that when $c_e(\overline{y})$ is true, execution may proceed from ℓ_e to $\tilde{\ell}_e$ while assigning the values $h_e(\overline{y})$ to the variables $\overline{y}$. Special cases are the semaphore instructions *request*(y) and *release*(y), equivalent to $(y > 0) \rightarrow [y := y - 1]$ and *true* $\rightarrow [y := y + 1]$, respectively. We refer the reader to [MP1] for a more detailed discussion of these models.

A program state for this system has the form:

$$\langle \ell^1, \ldots, \ell^m;\ \eta_1, \ldots, \eta_n \rangle,$$

where each $\ell^i \in L_i$ denotes the current location of the execution in the process P_i, and each $\eta_j \in D$ is the current value of the program variable y_j. (The variables $\overline{y}$ are assumed to range over some domain D.) Thus we identify the set of all states Σ as the set of all $(m+n)$-tuples $(L_1 \times \cdots \times L_m \times D^n)$.

The initiality predicate is given by:

$$\Theta(\ell^1, \ldots, \ell^m;\ \overline{y}):\quad [\bigwedge_{i=1}^{m} (\ell^i = \ell_0^i)] \ \wedge\ (\overline{y} = g(\overline{x}))$$

ensuring that all the processes are at their initial locations and the values of the program variables are properly initialized.

The set of transitions T is identified with the set of all edges $\bigcup_{i=1}^{m} E_i$. For $\tau = e \in E_i$ we define

$$(\tilde{\ell}^1, \ldots, \tilde{\ell}^m;\ \tilde{\overline{\eta}}) \ \in\ f_\tau(\ell^1, \ldots, \ell^m;\ \overline{\eta})$$

if and only if

$$\ell^i = \ell_e,\quad \tilde{\ell}^i = \tilde{\ell}_e,\quad \tilde{\ell}^j = \ell^j \text{ for every } j \neq i,\quad c_e(\overline{\eta}) = \textit{true} \quad \text{and} \quad \tilde{\overline{\eta}} = h_e(\overline{\eta}).$$

The justice family is given by:

$$\mathcal{J} \ =\ \{E_1, \ldots, E_m\};$$

that is, we require that justice be applied to each *process* individually. This implies that in any infinite computation, each process that has not terminated yet will eventually be scheduled.

The fairness family is given by:

$$\mathcal{F} \;=\; \{\{e\} \mid e \text{ is labelled by a } \textit{request}(y) \text{ instruction}\}.$$

Thus, each semaphore transition is to be individually treated fairly. This implies that a *request*(y) instruction which is waiting while y turns positive infinitely many times must eventually be performed.

In considering computations of a program as models for temporal formulas that express properties of the program, we define the model $\tilde{\sigma}$ corresponding to a sequence σ,

$$\sigma: \quad s_0 \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} s_2 \xrightarrow{\tau_3} \ldots,$$

as follows: If σ is infinite then the corresponding model is

$$\tilde{\sigma}: \quad s_0,\ s_1,\ s_2,\ \ldots.$$

In the case that σ is finite and its last state is the terminal state s_k, we take $\tilde{\sigma}$ to be

$$\tilde{\sigma}: \quad s_0,\ s_1,\ \ldots, s_k,\ s_k,\ \ldots\ ,$$

that is, the last state repeats forever.

3. THE PROOF SYSTEM

The proof system consists of three parts.

- Part A, called the *general part*, formalizes the pure temporal logic properties of sequences in general. It is completely independent of the particular program analyzed.
- Part B, called the *domain-dependent* part, formalizes the properties of the domain over which the program operates, such as integers, reals, strings, lists, trees, etc.
- Part C is the *program-dependent* part. It provides a formalization of the properties that result from restricting our attention to the computational sequences of the particular program being analyzed.

We refer the reader to [MP4], [MP5] for a discussion of parts A and B. Here we only repeat part C which we further develop in order to prove precedence properties.

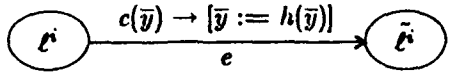
The program-dependent part consists of four axiom schemes corresponding to the four requirements imposed on admissible computations. In the following, a *state formula* is a formula containing no temporal operators and hence interpretable on a single state.

Let φ and ψ be two state formulas. We say that a transition τ *leads from* φ *to* ψ if for every two states s and s' the following is true:

$$\varphi(s) \wedge \left(s' \in f_\tau(s)\right) \;\Rightarrow\; \psi(s').$$

Note that this formula is classical, i.e., contains no temporal operators and should be expressible and provable in the first-order theory over the domain.

For example, in the case of the shared-variables computation model a transition τ would correspond to an edge e in some process P_i:



so that the condition above is expressible as

$$\varphi(\ell^1, ..., \ell^i, ..., \ell^m;\ \overline{y}) \wedge c(\overline{y}) \quad \Rightarrow \quad \psi(\ell^1, ..., \tilde{\ell}^i, ..., \ell^m;\ h(\overline{y})).$$

Given a subset of transitions $T' \subset T$, we say that T' ***leads from*** φ ***to*** ψ if every transition $\tau \in T'$ leads from φ to ψ. If the full set T leads from φ to ψ, we also say that ***the program*** P ***leads from*** φ ***to*** ψ.

The state formula *Terminal*, characterizes the terminal states:

$$\mathit{Terminal}(s) \ = \ \bigwedge_{\tau \in T} (f_\tau(s) = \phi).$$

Also, for a subset T' of transitions, the state formula *Enabled* characterizes the enabled transitions in T':

$$\mathit{Enabled}(T')(s) \ = \ \bigvee_{\tau \in T'} [f_\tau(s) \neq \phi].$$

Both formulas are expressible by a quantifier-free first-order formula.

Following are the inference rules of the program part:

(INIT) For an arbitrary temporal formula w

$$\frac{\vdash\ \mathit{\theta} \supset \Box w}{\vdash\ \Box w}$$

This rule states that if w is an invariant for all initialized computations it is also an invariant for all admissible computations. This is because every admissible computation is a suffix of an initialized computation, and a property of the form $\Box w$ is hereditary from a sequence to all of its suffixes.

(TRNS) Let φ and ψ be two state formulas

$$\frac{\vdash\ \text{Every } \tau \in T \text{ leads from } \varphi \text{ to } \psi \qquad \vdash\ (\varphi \wedge \mathit{Terminal}) \supset \psi}{\vdash\ \varphi \supset \bigcirc \psi}$$

The first premise ensures that as long as at least one transition is enabled, then if the current state satisfies φ, the next state must satisfy ψ. The second premise handles the case that all

transitions are disabled, i.e., that of a terminal state. In a computation this means that no further action is possible and the next state is identical to the present. Hence this premise also ensures that in such a case the next state will satisfy ψ.

(JUST) Let φ and ψ be two state formulas, and $T^J \in \mathcal{J}$ a justice set

$$\vdash \text{Every } \tau \in T \text{ leads from } \varphi \text{ to } \varphi \vee \psi$$

$$\vdash \text{Every } \tau \in T^J \text{ leads from } \varphi \text{ to } \psi$$

$$\vdash [\varphi \wedge \Box \mathit{Enabled}(T^J)] \supset \varphi \mathcal{U} \psi$$

To justify this rule, consider a computation σ such that $\varphi \wedge \Box \mathit{Enabled}(T^J)$ holds for σ but $\varphi \mathcal{U} \psi$ does not hold. By the first premise, once φ holds it can only stop holding when ψ happens. Hence $\varphi \mathcal{U} \psi$ may fail to hold only if ψ never happens and φ is true forever. Since we assumed that T^J is continuously enabled on σ, some transition in T^J must eventually be activated, and this in a state satisfying φ. Hence, by the second premise, once this transition is activated, it achieves ψ, contrary to our assumption.

A similar rule applies to fairness:

(FAIR) Let φ and ψ be two state formulas, and $T^F \in \mathcal{F}$ a fairness s

$$\vdash \text{Every } \tau \in T \text{ leads from } \varphi \text{ to } \varphi \vee \psi$$

$$\vdash \text{Every } \tau \in T^F \text{ leads from } \varphi \text{ to } \psi$$

$$\vdash [\varphi \wedge \Box \Diamond \mathit{Enabled}(T^F)] \supset \varphi \mathcal{U} \psi$$

The justification is similar to that of the JUST rule.

In the following discussion we will consider computations only under the assumption of justice. This amounts to considering an empty fairness family $\mathcal{F} = \phi$. In the shared-variables computation system this means that we consider programs without semaphores. The reintroduction of fairness to the following analysis can be done in a straightforward manner.

In [MP5] the set of the rules above has been shown to be relatively complete. By this we mean that an arbitrary property which is valid for a given program, can be proved using these rules, provided the pure logic and domain dependent parts are strong enough to prove all valid properties. This result implies that the program dependent part is adequate for establishing all the properties that are true for admissible computations. However, while giving full generality, these rules do not provide specific guidance for proving properties of the three important classes that we have discussed: invariance, liveness and precedence.

We will proceed to develop derived rules, one for each class. These rules, while being derivable in the general system, have the advantage of being complete for their classes. By this we mean, that every valid property in the class can be proved using a *single* application of the proposed rule as the only temporal step. All the premises to the rule are first-order over the domain. Thus, for anyone who is interested only in proving properties of these classes, the respective rules are the only temporal proof rules he may ever need, dispensing for example with the general temporal logic part.

We will illustrate these rules on a single example -- an algorithm for mutual exclusion (Fig. 0) -- taken from [Pe]. The program consists of two concurrent processes, P_1 and P_2 that compete on the access to their critical regions, presented by ℓ_3 and m_3 respectively. Entry into the critical regions is expected to be exclusive, i.e., at no time can P_1 be at ℓ_3 while at the same time P_2 is at m_3. The processes communicate by means of the shared-variables y_1, y_2, t. Process P_i sets y_i $(i = 1, 2)$ to T whenever he is interested in entering his critical region. He then proceeds to set t to i. Following, he reaches a waiting state (ℓ_2 or m_2, respectively). There he waits until either $y_{\bar{i}} = F$ (here $\bar{i}$ is the competing process, i.e., $\bar{1} = 2$ and $\bar{2} = 1$) or $t = \bar{i}$. In the first case he infers that the competitor is not currently interested. In the second case he infers that $P_{\bar{i}}$ is interested but has arrived to his waiting state *after* P_i did, since $P_{\bar{i}}$ was the *last* to set t to $\bar{i}$. In any of these cases P_i enters his critical region. Once he finishes his business there he exits while setting y_i to F, indicating loss of interest in further entries for the present.

This description is of course intuitive and informal. The following discussions will provide more formal proofs of the correctness of the algorithm.

4. INVARIANCE PROPERTIES

A single rule which is complete for this class is:

(INV) -- Invariance Rule

Let φ and ψ be state properties

A. $\vdash \theta \supset \varphi$

B. $\vdash$ Every $\tau \in T$ leads from φ to φ

C. $\vdash \varphi \supset \psi$

$\vdash \Box\psi$

A slightly more elaborate rule can similarly be used to establish properties of the form $\varphi \supset \Box\psi$.

Since the rule is derivable from the INIT and TRNS rules above, it is certainly sound.

To argue that it is complete for properties of the form $\Box\psi$, let ψ be a state property such that $\Box\psi$ is true for all computations. Define the predicate:

$$Acc(s) = \{\text{There exists an initialized computation segment } s_0 \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_k} s_k = s\}.$$

Thus, $Acc(s)$ is true for a state s iff there exists an initialized computation having s as one of its states. We have defined $Acc(s)$ in words rather than by a formula; however, if the underlying domain is rich enough to contain, say, the integers, then this predicate is expressible by a first-order formula over the domain.

We now apply the INV rule with $\varphi = Acc$. Certainly $\theta \supset Acc$, since every state s_0 satisfying θ participates in a computation: $s_0 \rightarrow s_1 \rightarrow \ldots$. It is also easy to see that if s is accessible and $s' \in f_\tau(s)$ then s' is also accessible. This establishes premise B. Premise C says that every accessible state satisfies ψ, but this follows from our assumption that $\Box\psi$ is true on all admissible computations. Consequently the INV rule is always applicable.

Let us consider some invariance properties for the mutual exclusiion program (Fig. 0) presented above. I_0 : $\vdash \Box((t=1) \vee (t=2))$

Note that for this program

$$\theta : \quad at\,\ell_0 \wedge at\,m_0 \wedge [(y_1, y_2, t) = (F, F, 1)].$$

Take $\varphi = \psi = (t=1) \vee (t=2)$. It is easy to verify that $\theta \supset \varphi$ since θ implies $t=1$. Similarly by inspecting every transition we see that all of them maintain φ.

I_1 : $\vdash \Box(y_1 \equiv \ell_{1..3})$

The proposition $\ell_{1..3}$ is defined as $at\,\ell_1 \vee at\,\ell_2 \vee at\,\ell_3$, i.e., it holds whenever P_1 is somewhere in $\{\ell_1, \ell_2, \ell_3\}$. Potentially falsifying transitions are:

$\ell_0 \rightarrow \ell_1$: setting both y_1 and $\ell_{1..3}$ to T.

$\ell_3 \rightarrow \ell_0$: setting both y_1 and $\ell_{1..3}$ to F.

All other transitions do not modify either y_1 or $\ell_{1..3}$.

I_2 : $\vdash \Box(y_2 \equiv m_{1..3})$.

This property is symmetric to I_1.

I_3 : $\vdash \Box\{[\ell_2 \wedge \sim m_2] \supset (t=1)\}$.

Note that initially ℓ_2 (i.e., $at\,\ell_2$) is false so that the implication is true. Potentially falsifying transitions are:

$\ell_1 \rightarrow \ell_2$: sets t to 1.
$m_1 \rightarrow m_2$: makes $\sim m_2$ false.
$m_2 \rightarrow m_3$ while ℓ_2: by I_1, $y_1 = T$ so this transition is possible only when $t=1$.

All other transitions trivially maintain the invariant.

I_4 : $\vdash \Box\{[m_2 \wedge \sim \ell_2] \supset (t=2)\}$.

Can be shown in a similar way.

We may now obtain the invariant ensuring mutual exclusion:

I_5 : $\vdash \Box(\sim \ell_3 \vee \sim m_3)$.

It is certainly true initially. The potentially falsifying transitions of this invariant are:

$\ell_2 \rightarrow \ell_3$ while m_3: but then $y_2 = T$ (by I_2) and $t=1$ (by I_3), so that this transition is impossible.

$m_2 \rightarrow m_3$ while ℓ_3: impossible, because $y_1 = T$ (by I_1) and $t=2$ (by I_4).

Thus mutual exclusion has been formally proved.

5. LIVENESS PROPERTIES

We start by developing a proof rule which is more convenient to apply than the JUST rule.

(J-EVNT) — The Just Eventuality Rule

Let φ and ψ be two state formulas and T^J a justice set

A. $\vdash$ Every $\tau \in T$ leads from φ to $\varphi \vee \psi$

B. $\vdash$ Every $\tau \in T^J$ leads from φ to ψ

C. $\vdash \varphi \supset (\psi \vee \mathit{Enabled}(T^J))$

$$\vdash \varphi \supset \varphi \mathcal{U} \psi$$

A similar rule exists for fairness. The rule can easily be derived from the JUST rule since by premise C every computation having in it a φ which is not followed by a ψ, will have T^J continuously enabled. This by the JUST rule implies $\varphi \mathcal{U} \psi$.

Let us apply the EVNT rule to our sample mutual exclusion program (Fig. 0). Take for example,

$$\varphi = \varphi_1: \quad at\ell_2 \wedge atm_2 \wedge (t = 2) \wedge (y_1 = T) \wedge (y_2 = T)$$

$$\psi = \varphi_0: \quad at\ell_3$$

Clearly the only transitions enabled on a state satisfying φ_1 are $\ell_2 \rightarrow \ell_3$ and $m_2 \rightarrow m_2$. Consequently every transition leads from φ_1 to $\varphi_1 \vee \psi$. Taking T^J to be P_1, i.e., all transitions within P_1, we have premises A and B obviously satisfied. Also φ_1 implies that $\ell_2 \rightarrow \ell_3$ and hence P_1 is enabled. Thus we obtain $\vdash \varphi_1 \supset (\varphi_1 \mathcal{U} \varphi_0)$. From this we can certainly obtain

$$\vdash \varphi_1 \supset \Diamond \varphi_0$$

since $p \mathcal{U} q$ implies $\Diamond q$.

Next let us take

$$\varphi = \varphi_2: \quad at\ell_2 \wedge atm_1 \wedge (y_1 = T) \wedge (y_2 = T)$$

$$\psi = \varphi_1 \vee \varphi_0.$$

We now take T^J to be P_2. Certainly, the only transitions possibly enabled under φ_2 are $\ell_2 \rightarrow \ell_2$, $\ell_2 \rightarrow \ell_3$ and $m_1 \rightarrow m_2$. The first transition preserves φ_2. The second transition leads from φ_2 to φ_0. The third transition which is guaranteed to be enabled under φ_2, leads from φ_2 to φ_1. Thus every transition leads from φ_2 to $\varphi_1 \vee \varphi_0$. We conclude $\vdash \varphi_2 \supset \Diamond(\varphi_1 \vee \varphi_0)$. From this we may conclude by temporal reasoning and the previously established $\vdash \varphi_1 \supset \Diamond \varphi_0$ that

$$\vdash \varphi_2 \supset \Diamond \varphi_0.$$

We may proceed and define additional φ_j, $j = 3, \ldots, 6$, such that for each j, $\vdash \varphi_j \supset \Diamond(\bigvee_{k<j} \varphi_k)$ which eventually leads to $\vdash \varphi_j \supset \Diamond \varphi_0$. This proof strategy of constructing a finite chain of assertions, each eventually leading to an assertion of lower index can be summarized by:

(CHAIN) — The Chain Reasoning Proof Principle

Let $\varphi_0, \varphi_1, \ldots, \varphi_r$ be a sequence of state formulas.

A. $\vdash$ Every $\tau \in T$ leads from φ_i to $\bigvee_{j \leq i} \varphi_j$,

B. For every $i > 0$ there exists a justice set $T^J = T_i^J$ such that
$\vdash$ Every $\tau \in T_i^J$ leads from φ_i to $\bigvee_{j<i} \varphi_j$

C. For every $i > 0$ and T_i^J as above:
$\vdash \varphi_i \supset [(\bigvee_{j<i} \varphi_j) \vee \mathit{Enabled}(T_i^J)]$

$\vdash (\bigvee_{i=0}^{r} \varphi_i) \supset \Diamond \varphi_0$

The scheme of a proof according to the CHAIN principle is best presented in a form of a diagram. In this diagram we have a node for each φ_i. For each transition τ leading from a state satisfying φ_i to a state satisfying φ_j with $j \neq i$ (and hence by A, $j < i$) we draw an edge from φ_i to φ_j. This edge is labelled by the appropriate justice set to which the transition belongs. Edges belonging to the justice set which is known by premise C to be enabled in φ_i are drawn as double edges. For example, Fig. 1 contains a proof diagram for proving $\vdash at\,\ell_1 \supset \Diamond\, at\,\ell_3$ for the mutual exclusion program. By the CHAIN rule we actually proved $\vdash (\bigvee_{i=0}^{6} \varphi_i) \supset \Diamond\, at\,\ell_3$, but since φ_6 is $at\,\ell_1$ this establishes the desired result. The diagram representation of the CHAIN rule resembles closely the proof lattice advocated in [OL] for proving liveness properties.

In the application of the CHAIN rule we may freely use any previously derived invariances of the program. Thus, if $\vdash \Box I$ is any previously derived invariance, we may use $\varphi_i \wedge I$ instead of φ_i to establish any of the premises. This amounts to considering the sequence $\varphi_0 \wedge I, \ldots, \varphi_r \wedge I$ instead of the original sequence of assertions. Thus in the diagram (Fig. 1) we did not have an assertion corresponding to (ℓ_3, m_3) since by the previously established invariances such a situation is impossible, in particular no transition could lead from $I \wedge \varphi_4$ to (ℓ_3, m_3). Similarly no transition from (ℓ_2, m_1) to ℓ_3 has been drawn in view of I_3.

The chain reasoning principle assumed a finite number of links in the chain. It is quite adequate for finite state programs, i.e., programs where the variables range over finite domains. However, once we consider programs over infinite domains, such as the integers, it is no longer sufficient to consider only finitely many assertions. In fact, sets of assertions of quite high cardinality are needed. The obvious generalization to infinite sets of assertions is to consider a single state assertion $\varphi(\alpha, s)$, parametrized by a parameter α taken from a well-founded ordered set $(A, \prec)$. Obviously, an important feature of our chain of assertions is that program transitions led from φ_i to φ_j with $j < i$. This property can also be stated for an arbitrary well-founded ordering. Thus a natural generalization of the chain reasoning rule is the following:

(WELL) — The Well Founded Liveness Principle

Let $(\mathbb{A}, \prec)$ be a well-founded ordered set.

Let $\varphi(\alpha) = \varphi(\alpha, s)$ be a parametrized state formula, and ψ a state formula.

Let $h : \mathbb{A} \rightarrow \mathcal{J}$ be a helpfulness function identifying for each $\alpha \in \mathbb{A}$ the helpful justice set $h(\alpha) \in \mathcal{J}$.

A. $\vdash$ Every transition $\tau \in T$ leads from
$\varphi(\alpha)$ to $\psi \vee \exists\beta\big((\beta \preceq \alpha) \wedge \varphi(\beta)\big)$

B. $\vdash$ Every transition $\tau \in h(\alpha)$ leads from
$\varphi(\alpha)$ to $\psi \vee \exists\beta\big((\beta \prec \alpha) \wedge \varphi(\beta)\big)$

C. $\vdash \varphi(\alpha) \supset [\psi \vee \exists\beta\big((\beta \prec \alpha) \wedge \varphi(\beta)\big) \vee \mathit{Enabled}\big(h(\alpha)\big)]$

$\vdash (\exists\alpha.\varphi(\alpha)) \supset \Diamond\psi$

In order to obtain a complete rule for liveness properties we have to treat the parametrized assertion $\varphi(\alpha, s)$ as an auxiliary assertion:

(LIVE) — A Complete Principle for Liveness

Let p, q be state formulas and $\varphi(\alpha)$, ψ a parametrized assertion pair as in WELL.

Assume premises A, B, C as in WELL, and

D. $\vdash \Box p$, i.e., p is an invariant

E. $\vdash (q \wedge p) \supset (\exists\alpha.\varphi(\alpha))$

$\vdash q \supset \Diamond\psi$

We refer the reader to [LPS] for a completeness proof of the LIVE principle. Completeness here means that given two state properties q and ψ such that $q \supset \Diamond\psi$ is a valid statement over all the computations of the program P, it is always possible to find state predicates p, $\varphi(\alpha, s)$ with $\alpha \in \mathbb{A}$ and $(\mathbb{A}, \prec)$, h as in WELL that satisfy premises A to E. Note that premise D requires preliminary derivation of the invariance of p which can be done using the INV rule.

6. PRECEDENCE PROPERTIES

As a key operator in expressing and establishing precedence properties we take the weak until operator, $\mathcal{U}$, to which we will refer here as the ***unless operator***.

The *unless* operator may be defined in terms of the standard *until* operator as:

$$p\,\mathcal{U}\,q \;\equiv\; \Box p \vee (p\,\mathcal{U}\,q).$$

Thus, in contrast to $p\mathcal{U}q$ it does not require that q eventually happen. But in the case that q never happens p is required to hold forever.

Even though it is introduced here as a derived operator, it can be adopted as the basic operator for establishing precedence properties. This is because both the *until* and *precede* operators can be expressed in terms of the *unless* operator:

$$p \,\mathcal{U}\, q \;\equiv\; (p \,\mathfrak{U}\, q) \wedge \Diamond q$$

$$p \,\mathcal{P}\, q \;\equiv\; (\sim q) \,\mathfrak{U}\, (p \wedge \sim q).$$

We can also express the nested *until* operator by considering the nested *unless* operator. Let $\psi_r, \psi_{r-1}, \ldots, \psi_1, \psi_0$ be a sequence of formulas then

$$\psi_r \,\mathfrak{U}\, \psi_{r-1} \,\mathfrak{U}\, \ldots \psi_1 \,\mathfrak{U}\, \psi_0 \;\equiv\; \psi_r \,\mathfrak{U}\, (\psi_{r-1} \,\mathfrak{U}\, (\ldots (\psi_1 \,\mathfrak{U}\, \psi_0))\ldots)$$

holds on a sequence $\sigma = s_0, s_1, \ldots$ if there exists a sequence of indices $0 = i_r \leq i_{r-1} \leq \ldots \leq i_1 \leq i_0 \leq \omega$ such that for every $\ell > 0$ and j, $i_\ell \leq j < i_{\ell-1}$, ψ_ℓ holds on

$$\sigma^{(j)} \;=\; s_j, s_{j+1}, \ldots$$

and if $i_0 < \omega$ then ψ_0 holds on $\sigma^{(i_0)}$. Note that some of the i_ℓ may be equal to one another, and also to ω in which case some of the ψ_ℓ hold in empty periods.

An alternative description is that $\psi_r \,\mathfrak{U}\, \ldots \psi_1 \,\mathfrak{U}\, \psi_0$ holds on σ iff either σ satisfies $\psi_r \mathcal{U} \ldots \psi_1 \mathcal{U} \psi_0$ or for some j, $0 < j \leq r$, σ satisfies $\psi_r \mathcal{U} \ldots \psi_{j+1} \mathcal{U} \Box \psi_j$. In the case $j = r$, σ satisfies $\Box \psi_r$.

Then we can express the nested *until* by an extension of the previous formula for a simple *until*:

$$\psi_r \mathcal{U} \psi_{r-1} \mathcal{U} \ldots \psi_1 \mathcal{U} \psi_0 \;\equiv\; (\psi_r \,\mathfrak{U}\, \psi_{r-1} \,\mathfrak{U}\, \ldots \psi_1 \,\mathfrak{U}\, \psi_0) \wedge \Diamond \psi_0.$$

Let us justify this equivalence. The direction in which the nested *until* implies the nested *unless* and the eventual ocurrence of ψ_0 is obvious. Let us therefore consider the other direction.

Assume that $\psi_r \,\mathfrak{U}\, \ldots \psi_1 \,\mathfrak{U}\, \psi_0$ and $\Diamond \psi_0$ both hold on a sequence σ. By the interpretation of nested *unless* there exists a partition:

$$0 = i_r \leq i_{r-1} \leq \ldots \leq i_1 \leq i_0 \leq \omega$$

such that ψ_ℓ holds between i_ℓ and $i_{\ell-1}$ for $\ell > 0$ and ψ_0 holds at i_0 if it is finite. Since ψ_0 must occur somewhere in σ let j be the minimal index such that ψ_0 holds on $\sigma^{(j)}$. If $j = i_0 < \omega$, then the same partition justifies $\psi_r \mathcal{U} \ldots \psi_1 \mathcal{U} \psi_0$ on σ. Otherwise there exists some ℓ such that $i_\ell \leq j < i_{\ell-1}$. In this case the partition up to i_ℓ and then j justifies $\psi_r \mathcal{U} \ldots \psi_\ell \mathcal{U} \psi_0$ from which

$$\psi_r \mathcal{U} \ldots \psi_\ell \mathcal{U} \psi_{\ell-1} \ldots \psi_1 \mathcal{U} \psi_0$$

follows by letting $\psi_{\ell-1}, \ldots, \psi_1$ hold over empty periods.

Thus, expressively at least, the *unless* operator seems to be an appropriately basic operator. But we claim that the choice of the *unless* operator is appropriate on proof theoretic grounds as well. By inspecting the expression of *until* formulas in terms of *unless* formulas we find a resemblance

to the relation between the concepts of total and partial correctness. Total correctness, which is a liveness property, can be expressed as the conjunction of partial correctness, which is an invariance property, and termination, which is another liveness property but simpler than the original. In quite the same way we can express the *until* property as a conjunction of an *unless* property, which we regard as extended invariance property and the simpler liveness property $\Diamond \psi_0$.

In practice, if we want a single proof principle that will cover properties of the following three subclasses

$$(a) \quad \varphi \supset (p \,\mathcal{U}\, q)$$

$$(b) \quad \varphi \supset (p \,\mathcal{P}\, q)$$

$$(c) \quad \varphi \supset (p \,\mathfrak{U}\, q)$$

then the *unless* operator is a good choice.

In order to establish (a) we establish separately

$$\vdash (\varphi \supset p \,\mathfrak{U}\, q) \qquad \text{and} \qquad \vdash \varphi \supset \Diamond q,$$

which are implied by (a). The first will be established by using the *unless* proof principle. The second is a liveness property and can be established by the WELL rule or its extensions.

Similarly in order to establish (b) it is sufficient to establish $\varphi \supset (\tilde{p} \,\mathfrak{U}\, \tilde{q})$ where $\tilde{p}$ is $\sim q$ and $\tilde{q}$ is $p \wedge \sim q$.

We could not have used the *until* operator in a similar role, i.e., reducing proofs of properties of the subclasses (b) and (c) to those of (a). This is for example because if $\varphi \supset (p \,\mathfrak{U}\, q)$ is a valid statement, then certainly so is $\varphi \supset \big(\Box p \vee (p \mathcal{U} q)\big)$, but it does not imply that either $\varphi \supset \Box p$ or $\varphi \supset (p \mathcal{U} q)$ are valid statements. Proving *precede* statements would cause similar problems.

The fact that the weak form of the *until* operator is more basic than its strong form seems to have been intuitively sensed in [L2] where a *while* operator is introduced which is equivalent to $p \,\mathfrak{U} \sim q$.

Consequently, we will proceed by developing proof principles for the *unless* operator $\mathfrak{U}$. We begin by formulating a core rule:

(CORE-U) — Core Rule for Unless Properties

Let $\varphi_r, \varphi_{r-1}, \ldots, \varphi_0$ be state formulas

A. For every $i > 0$,

$\vdash$ Every $\tau \in T$ leads from φ_i to $\bigvee_{j \le i} \varphi_j$

$$\vdash \Big(\bigvee_{i=0}^{r} \varphi_i\Big) \supset (\varphi_r \,\mathfrak{U}\, \varphi_{r-1} \,\mathfrak{U}\, \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0)$$

Let σ be a computation whose first state s_0 satisfies φ_j for some $0 \le j \le r$. Assume first that $j > 0$. Define $i_r = i_{r-1} = \ldots = i_j = 0$. By premise A, s_1 must satisfy some φ_ℓ for $\ell \le j$. If

$\ell = j$ we proceed until we find an s_k that satisfies φ_ℓ for $\ell < j$. If we never find such a state we may take $i_{j-1} = \ldots = i_0 = \omega$. Otherwise we take $i_{j-1} = \ldots = i_\ell = k$ and proceed similarly beyond s_k unless $\ell = 0$. This construction shows that if s_0 satisfies φ_j for some j then σ satisfies $\varphi_r \,\mathfrak{U} \ldots \mathfrak{U}\, \varphi_0$. The case $j = 0$ is even simpler.

We can make a complete rule out of the CORE-U rule by strengthening the preconditions and weakening the post conditions.

(UNLS) — Complete Rule for Unless Properties

Let $\varphi_r, \ldots, \varphi_0, \psi_r, \ldots, \psi_0, p, q$ be state formulas such that:

A. For every $i > 0$,

$\vdash$ Every $\tau \in T$ leads from $\varphi_i \wedge p$ to $\bigvee_{j \le i} \varphi_j$

B. $\vdash \Box p$

C. $\vdash (q \wedge p) \supset (\bigvee_{i=0}^{r} \varphi_i)$

D. For every i, $0 \le i \le r$

$\vdash (\varphi_i \wedge p) \supset \psi_i$

$\vdash q \supset (\psi_r \,\mathfrak{U}\, \psi_{r-1} \,\mathfrak{U} \ldots \psi_1 \,\mathfrak{U}\, \psi_0)$

Let us consider the application of this rule to the analysis of the mutual exclusion algorithm. We take (the φ_i's refer to the assertions in Fig. 1):

q: $\quad at\, \ell_2$

$\tilde{\varphi}_0 = \psi_0$: $\quad at\, \ell_3$

$\tilde{\varphi}_1 = \varphi_{1..3}$: $\quad \ell_2 \wedge [m_{0,1} \vee (m_2 \wedge (t = 2))]$

$\tilde{\varphi}_2 = \varphi_4$: $\quad \ell_2 \wedge m_3$

$\tilde{\varphi}_3 = \varphi_5$: $\quad \ell_2 \wedge m_2 \wedge (t = 1)$

$\psi_1 = \psi_3 = \sim m_3, \qquad \psi_2 = m_3$

p — the conjunction of all the invariants $I_0 \wedge \ldots \wedge I_5$

The diagram certainly establishes that $\tilde{\varphi}_i$, $i > 0$, leads to $\bigvee_{j \le i} \tilde{\varphi}_j$.

It is also easy to show that $(q \wedge p) \supset (\bigvee_{i=1}^{3} \tilde{\varphi}_i)$ and that $\tilde{\varphi}_i \supset \psi_i$ for $i = 0, \ldots, 3$. Thus we may conclude:

$$\vdash \ell_2 \supset (\sim m_3 \,\mathfrak{U}\, m_3 \,\mathfrak{U} \sim m_3 \,\mathfrak{U}\, \ell_3).$$

This establishes the property of 1-bounded overtaking from ℓ_2. This means that once P_1 is at ℓ_2, P_2 may be at m_3 at most once before P_1 gets to his critical section at ℓ_3.

An alternative derivation of the same result could have been achieved by taking the φ's in the rule to be identical to the φ's in the diagram. This leads to:

$$\vdash \quad \ell_2 \supset (\varphi_5 \,\mathfrak{U}\, \varphi_4 \,\mathfrak{U}\, \varphi_3 \,\mathfrak{U}\, \varphi_2 \,\mathfrak{U}\, \varphi_1 \,\mathfrak{U}\, \varphi_0).$$

We may now use the collapsing theorem for the *unless* operator:

$$(p \,\mathfrak{U}\, q \,\mathfrak{U}\, r) \supset ((p \vee q) \,\mathfrak{U}\, r)$$

to obtain:

$$\vdash \quad \ell_2 \supset (\varphi_5 \,\mathfrak{U}\, \varphi_4 \,\mathfrak{U}\, (\varphi_1 \vee \varphi_2 \vee \varphi_3) \,\mathfrak{U}\, \varphi_0),$$

which is equivalent to the above after we replace each of the φ_i's by the weaker ψ_i.

Having obtained 1-bounded overtaking from the point that P_1 is at ℓ_2 we may inquire whether the same holds from the point that P_1 is at ℓ_1. As the analysis shows in Fig. 2 the best we can hope for is 2-bounded overtaking. The diagram in Fig. 2 establishes

$$\vdash \quad \ell_1 \supset (\varphi_8 \,\mathfrak{U}\, \varphi_{5..7} \,\mathfrak{U}\, \varphi_4 \,\mathfrak{U}\, \varphi_{1..3} \,\mathfrak{U}\, \varphi_0)$$

from which 2-bounded overtaking is easily established.

7. COMPLETENESS OF THE UNLS RULE

Next we will show that the UNLS rule presented above is complete for establishing nested *unless* properties.

Proof:

Let $q, \psi_r, \ldots, \psi_0$ be state properties such that the statement $q \supset (\psi_r \,\mathfrak{U}\, \psi_{r-1} \ldots \psi_1 \,\mathfrak{U}\, \psi_0)$ is valid on all admissible computations. We will show that there exist state properties $p, \varphi_r, \ldots, \varphi_0$, which are first-order expressible over the integers, such that all the premises of the UNLS rule are satisfied.

As p we choose

$$p(s) \;\equiv\; Acc(s) \;\equiv\; \{\text{There exists an initialized computation containing } s\}.$$

Clearly p is an invariant of all admissible computations so that premise B is satisfied.

Let $\tilde{\sigma}$ be a finite segment of a computation, i.e., a finite sequence

$$\tilde{\sigma} \;=\; s_0 \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_k} s_k$$

such that $s_{i+1} \in f_\tau(s_i)$ for each $i = 0, \ldots, k-1$.

We say that $\tilde{\sigma}$ satisfies a temporal formula w if $\tilde{\sigma}$'s infinite extension $s_0, s_1, \ldots, s_k, s_k, s_k, \ldots$ satisfies w.

Let σ be a computation satisfying $\psi_r \,\mathfrak{U} \ldots \psi_1 \,\mathfrak{U}\, \psi_0$. It can be verified that any finite prefix of σ is a computation segment that also satisfies $\psi_r \,\mathfrak{U} \ldots \psi_1 \,\mathfrak{U}\, \psi_0$.

Let us define now φ_i for $i = 0, 1, \ldots, r$ by $\varphi_i(s) = \mathit{true}$ iff

(*a*) Every computation segment originating at s satisfies $\psi_i \,\mathfrak{U}\, \psi_{i-1} \ldots \psi_1 \,\mathfrak{U}\, \psi_0$

(*b*) The index i is the smallest index for which (*a*) holds.

Let us show that the sequence of φ_i's defined in this way satisfies premises A, C and D of the UNLS rule.

Consider first premise A. Let s be a state satisfying φ_i, for $i > 0$. Let s' be a state such that $s' \in f_\tau(s)$. Consider any computation segment originating in s':

$$\tilde{\sigma}' : \quad s' \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_k} s_k.$$

We can obtain from it a computation segment:

$$\tilde{\sigma} : \quad s \xrightarrow{\tau} s' \xrightarrow{\tau_1} s_1 \xrightarrow{\tau_2} \ldots \xrightarrow{\tau_k} s_k.$$

By our assumption about s, $\tilde{\sigma}$ must satisfy $\psi_i \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$. It can be shown that due to $i > 0$, and the minimality of i this implies that $\tilde{\sigma}'$ must also satisfy $\psi_i \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$. Thus we have identified at least one index, i, such that clause (*a*) is satisfied for i and s'. Let $j \geq 0$ now be the minimal index satisfying (*a*) for s'. Then (*b*) is also satisfied and we have that s' satisfies φ_j for $j \leq i$. This establishes premise A.

Next, consider premise C. Let s be a state satisfying q and p. It is therefore an accessible state satisfying q. By the assumption that $q \supset (\psi_r \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0)$ is a valid statement for all admissible computations, every computation originating in s saisfies $\psi_r \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$. Consequently every computation segment originating in s satisfies $\psi_r \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$. Thus, clause (*a*) of the definition of φ_i is satisfied for $i = r$. Let j be the minimal index satisfying clause (*a*). Then $\varphi_j(s)$ holds and $j \leq r$.

To show premise D, let s be a state saisfying φ_i. Consider first $i = 0$. The zero version of $\psi_i \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$ is ψ_0 by itself. Since every finite computation segment originating in s must saisfy ψ_0 which is a state property, it follows that s satisfies ψ_0. Consider next, $i > 0$. Since i was the minimal index satisfying clause (*a*), there must exist a computation segment σ originating in s which satisfies $\psi_i \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$ but not $\psi_{i-1} \,\mathfrak{U} \ldots \,\mathfrak{U}\, \psi_0$. Consequently the initial section of $\tilde{\sigma}$ satisfying ψ_i must be non-empty and therefore s must satisfy ψ_i. Thus, we have $\varphi_i \supset \psi_i$.

We claimed that the φ_i's defined above are first-order expressible over the integers. This is due to the fact that clause (*a*) refers only to *finite* computation segments. This is a direct consequence of the fact that we deal with the *unless* operator. No similar first-order definition is possible for the *until* operator. ∎

8. DIRECT PROOFS OF UNTIL PROPERTIES

In spite of our recommendation of splitting a proof of *until* property into a proof of a similar *unless* property, followed by a liveness proof of $\Diamond \psi$, there are many cases in which an *until* property can be directly obtained by a small modification of the liveness proof. As we have seen both the CHAIN rule and the UNLS rule call for a sequence of assertions, such that the computation always lead from φ_i to φ_j with $j \leq i$. The CHAIN rule stipulates in addition a strict decrease under certain conditions. It is often the case that the same chain of assertions used in the CHAIN rule can be used to establish a nested *until*. In fact, in much the same way that we have justified the CHAIN rule we can with the same premises obtain a stronger result:

Taking $0 < p_1 < p_2 < \ldots < p_s = r$ be a partition of the index range $[0...r]$ into s contiguous segments, we may formulate the following chain principle for *until* properties:

(U-CHAIN) — The Chain Rule for Until Properties

Let $\varphi_0, \varphi_1, \ldots, \varphi_r$ be a sequence of state formulas, and $0 < p_1 < p_2 < \ldots < p_s = r$ a partition of $[1...r]$.

A. $\vdash$ Every $\tau \in T$ leads from φ_i to $(\bigvee_{j \leq i} \varphi_j)$ for $i = 1, \ldots, r$.

B. for every $i > 0$ there exists a justice set $T^J = T_i^J$ such that:
$\vdash$ Every $\tau \in T_i^J$ leads from φ_i to $(\bigvee_{j<i} \varphi_j)$

C. for $i > 0$ and T_i^J as above:
$\vdash \varphi_i \supset [(\bigvee_{j<i} \varphi_j) \vee \mathit{Enabled}(T_i^J)]$

$$\vdash \left(\bigvee_{i=0}^{r} \varphi_i\right) \supset \left[\left(\bigvee_{j=p_{s-1}+1}^{p_s} \varphi_j\right) \mathcal{U} \left(\bigvee_{j=P_{s-2}+1}^{p_{s-1}} \varphi_j\right) \mathcal{U} \ldots \left(\bigvee_{j=1}^{p_1} \varphi_j\right) \mathcal{U} \varphi_0\right]$$

The conclusion states that starting at a state that satisfies one of the φ_i's, $i = 0, \ldots, r$, we are guaranteed to have a period in which $(\bigvee_{j=p_{s-1}+1}^{p_s} \varphi_j)$ continuously hold, followed by a period in which $(\bigvee_{j=p_{s-2}+1}^{p_{s-1}} \varphi_j)$ continously holds, etc., until finally φ_0 is realized. Any of these periods may be empty.

To justify the soundness of this conclusion we first prove it for the most refined partition possible, namely:

$$(*) \qquad \left(\bigvee_{i=0}^{r} \varphi_i\right) \supset (\varphi_r \mathcal{U} \varphi_{r-1} \mathcal{U} \varphi_{r-2} \mathcal{U} \ldots \varphi_1 \mathcal{U} \varphi_0).$$

This is proved in a way similar to the justification of the corresponding liveness principle. We show

by induction on n, $n = 0, 1, \ldots, r$, that

$$\vdash \left(\bigvee_{i=0}^{n} \varphi_i\right) \supset (\varphi_n \mathcal{U} \varphi_{n-1} \mathcal{U} \ldots \varphi_1 \mathcal{U} \varphi_0).$$

For $n = 0$ we have $\vdash \varphi_0 \supset \varphi_0$ from which follows trivially

$$\vdash \varphi_0 \supset \varphi_0 \mathcal{U} \varphi_0.$$

Assume that the statement $(*)$ above has been proved for a certain n and consider its proof for $n + 1$.

Consider the EVNT rule with $\varphi = \varphi_{n+1}$, $\psi = \left(\bigvee_{i=1}^{n} \varphi_i\right)$. As shown in the proof of the liveness case all the premises of the EVNT rule are satisfied. Consequently we may conclude:

$$\vdash \varphi_{n+1} \supset \varphi_{n+1} \mathcal{U} \left(\bigvee_{i=1}^{n} \varphi_i\right).$$

By the induction hypothesis and the monotonicity of the $\mathcal{U}$ operator this yields

$$\vdash \varphi_{n+1} \supset (\varphi_{n+1} \mathcal{U} \varphi_n \mathcal{U} \ldots \varphi_1 \mathcal{U} \varphi_0).$$

Due to $\vdash v \supset (u \mathcal{U} v)$, the induction hypothesis can also be written as

$$\vdash \left(\bigvee_{i=0}^{n} \varphi_i\right) \supset (\varphi_{n+1} \mathcal{U} \varphi_n \mathcal{U} \ldots \varphi_1 \mathcal{U} \varphi_0).$$

Taking the disjunction of the last two statements gives

$$\vdash \left(\bigvee_{i=0}^{n+1} \varphi_i\right) \supset (\varphi_{n+1} \mathcal{U} \varphi_n \mathcal{U} \ldots \varphi_1 \mathcal{U} \varphi_0),$$

which is the required statement $(*)$ for $n + 1$.

Consider now a coarser partition:

$$0 < p_1 < p_2 < \ldots < p_s = r.$$

By consecutively merging any two contiguous assertions that fall into the same cell, using the collapsing rule:

$$\vdash (\varphi_{i+1} \mathcal{U} (\varphi_i \mathcal{U} \varphi)) \supset ((\varphi_{i+1} \vee \varphi_i) \mathcal{U} \varphi),$$

we obtain the coarser conclusion:

$$\vdash \left(\bigvee_{i=0}^{r} \varphi_i\right) \supset \left(\left(\bigvee_{j=p_{s-1}+1}^{p_s} \varphi_j\right) \mathcal{U} \left(\bigvee_{j=p_{s-2}+1}^{p_{s-1}} \varphi_j\right) \mathcal{U} \ldots \left(\bigvee_{j=1}^{p_1} \varphi_j\right) \mathcal{U} \varphi_0\right). \quad \blacksquare$$

In our mutual exclusion program, by reference to Fig. 1 it is easy to use the U-CHAIN rule and obtain:

$$\ell_2 \supset (\varphi_5 \mathcal{U} \varphi_4 \mathcal{U} \varphi_{1..3} \mathcal{U} \varphi_0),$$

from which the 1-bounded overtaking from ℓ_2 is obtained by the monotonicity of the *until* operator (i.e., replacing formulas by weaker formulas).

A natural extension of the U-CHAIN rule to programs that require infinite chains of assertions uses again well-founded ordered sets.

Let $(\mathcal{A}, \prec)$ be a well-founded ordered set. We require however that the ordering is total (or linear). That is, for every two distinct elements, $\alpha_1, \alpha_2 \in \mathcal{A}$ either $\alpha_1 \prec \alpha_2$ or $\alpha_2 \prec \alpha_1$.

(U-WELL) — Well-Founded Until Rule

Let $(\mathcal{A}, \prec)$ be a well-founded totally ordered set.

Let $\varphi(\alpha) = \varphi(\alpha, s)$ be a parametrized state formula.

Let $h : \mathcal{A} \to \mathcal{J}$ be a helpfulness function identifying for each $\alpha \in \mathcal{A}$ the helpful justice set $h(\alpha) \in \mathcal{J}$.

Let $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_s$ be a *finite* sequence of elements of $\mathcal{A}$.

A. $\vdash$ Every transition $\tau \in T$ leads from
$\varphi(\alpha)$ to $\psi \vee \exists\beta\big((\beta \preceq \alpha) \wedge \varphi(\beta)\big)$

B. $\vdash$ Every transition $\tau \in h(\alpha)$ leads from
$\varphi(\alpha)$ to $\psi \vee \exists\beta\big((\beta \prec \alpha) \wedge \varphi(\beta)\big)$

C. $\vdash \varphi(\alpha) \supset [\psi \vee \exists\beta\big((\beta \prec \alpha) \wedge \varphi(\beta)\big) \vee \mathit{Enabled}(h(\alpha))]$

$$\vdash \exists\alpha\big((\alpha \preceq \alpha_s) \wedge \varphi(\alpha)\big) \supset \big[\exists\beta\big((\alpha_{s-1} \prec \beta \preceq \alpha_s) \wedge \varphi(\beta)\big) \mathcal{U} \exists\beta\big((\alpha_{s-2} \prec \beta \preceq \alpha_{s-1}) \wedge \varphi(\beta)\big) \mathcal{U} \ldots \exists\beta\big((\beta \preceq \alpha_1) \wedge \varphi(\beta)\big) \mathcal{U} \psi\big]$$

By a combination of the completeness of the WELL rule for liveness properties and the UNLS rule for *unless* properties we can extend the above rule to a complete rule for *until* properties.

9. DECISION PROCEDURES FOR FINITE STATE PROGRAMS

The question of whether a given program has a certain property expressed by a temporal formula, is in general highly undecidable. However, for a very important restricted class of programs, this question is decidable, namely for finite state programs. Finite state programs are programs whose variables range each over a finite domain. These programs generate only finitely many different states and a joint finite transition diagram over these states can be constructed such that any computation is a maximal path in this finite directed graph. The literature abounds in many special decision procedures for testing for deadlock situations, starvation, etc. on programs

represented by finite transition diagrams. All these are special cases of the general result which states that testing a temporal formula over a finite state program is decidable. The general decision procedure for testing a temporal formula φ on a finite state program P consists in checking the implication $W_P \supset \varphi$ for general validity. In this implication W_P is a formula characterizing all admissible computations of P. If P is finite state then both W_P and φ may be represented as ***propositional*** temporal formulas. Consequently we test a propositional temporal formula for general validity. As shown in [PS], it can be done in time exponential in the size of P and φ. This exponential time complexity has been a source of criticism of linear temporal logic in [CES].

In this section we show that when the temporal property φ to be tested, falls into one of the property classes discussed here, then there exists an efficient decision procedure polynomial in the size of P and φ for testing φ on P.

Let P be a program consisting of m processes $P_1, \ldots, P_m$. Let each process P_i be presented as transition diagram with set of nodes L_i. The program variables $y_1, \ldots, y_n$ assume values over finite domains $D_1, \ldots, D_n$ respetively. Then the state set S of the program P is the set of all possible tuples $\langle \ell_1, \ldots, \ell_m; \eta_1, \ldots, \eta_n \rangle$ with $\ell_i \in \mathsf{L}_i$, $i = 1, \ldots, m$, and $\eta_j \in D_j$ for $j = 1, \ldots, n$. Consequently

$$|\mathsf{S}| \leq |\mathsf{L}_1| \times \cdots \times |\mathsf{L}_m| \times |D_1| \times \cdots \times |D_n|.$$

We construct for P a joint transition diagram T_P with S as nodes, and an edge $s \xrightarrow{P_i} s'$ for every pair of states s, s' and a transition τ in P_i which leads from s to s'.

In order to generate only accessible states we start from all states satisfying θ and include in T_P only states which are derivable from states which are already included in T_P. Fig. 3 shows the diagram T_P for the mutual exclusion algorithm. States in this diagram have the form (ℓ_i, m_j, t). We have not included the values of y_1, y_2 since in all accessible states they are uniquely determined by the location values ℓ_i and m_j. The initial state in this diagram is s_0.

We proceed to describe three algorithms which, for properties in each of the three classes, will determine whether a finite state program P has this property. The algorithms will be linear in the size of T_P. Let us denote $N = |T_P|$.

10. TESTING INVARIANCES

Let the formula to be tested be of the form $q \supset \Box\varphi$. We can check whether all paths in T_P, and hence all admissible computations of P, satisfy $q \supset \Box\varphi$ by the following procedure:

PI: Locate in T_P all states which satisfy q. For each such state s construct the transition diagram $T_P(s)$ which includes exactly all the states accessible from s. Check that each $s' \in T_P(s)$ satisfies φ.

If all these steps succeeded then $q \supset \Box\varphi$ is valid for P. We can organize the procedure so that it takes no more than $m \cdot N$ steps where $N = |T_P|$ and m is the number of processes and hence the maximal degree of T_P. This is because if $s_2 \in T_P(s_1)$ satisfies q then $T_P(s_2) \subset T_P(s_1)$ and no separate check is needed for s_2 if we have already checked $T_P(s_1)$.

Consequently we have to access each state at most once, and then may have to explore each of its edges.

For checking invariances we may actually suggest a simpler procedure: mark in T_P each state which is accessible from a q-state (a state saisfying q). Then check that all the marked s ates satisfy φ. However the complexity of the two procedures is identical and the PI procedure above conforms better with the procedures presented below for the other classes.

We may for example apply PI to test for the invariance of I_0 to I_5 derived for the mutual exclusion. All these properties have the form $\Box\varphi$ so we may take $q = \mathit{true}$ and consider $T_P(s)$ for all accessible states. However since every accessible state $s \in T_P(s_0) = T_P$, it is sufficient to check that all states in T_P satisfy φ.

Indeed we can easily check for example that there are no states in which ℓ_2, $\sim m_2$ and $t \neq 1$ are all true. In other words every state in which both ℓ_2 and $\sim m_2$ are true, i.e., s_6, s_{19}, also has $t = 1$ in it. This establishes I_3. Similarly, there is no accessible state in which both ℓ_3 and m_3 hold, establishing I_5.

It is easy to prove:

Lemma:

A formula $q \supset \Box\varphi$ is valid for P iff the procedure PI applied to T_P succeeds.

11. TESTING LIVENESS

Let the formula to be tested be of the form $q \supset \Diamond\varphi$. Let $s \in T_P$ be an accessible state. Let $\pi = s_1, \ldots, s_k$ be a finite path in T_P. We say that π is a non-φ path if none of $s_1, \ldots, s_{k-1}$ satisfy φ. Note that s_k is allowd to satisfy φ. We define $T_P(s, \varphi)$ to be the directed graph containing all states in T_P which are accessible from s by non-φ paths. The graph $T_P(s, \varphi)$ can be efficiently constructed as follows:

(*a*) Put s in $T_P(s, \varphi)$

(*b*) For every $s' \in T_P(s, \varphi)$ which does not satisfy φ, add all the successors of s' to $T_P(s, \varphi)$.

Let us decompose $T_P(s, \varphi)$ into maximal strongly connected components. It is known that when we consider edges between the components, it is always possible to order the components in a topological sorting order $K_1, \ldots, K_r$, such that if there is an edge from a node in K_i to a node in K_j then necessarily $i \leq j$. Components such that there are no edges leading out of them are called *terminal* components.

We suggest the following test for checking that all just computations in $T_P(s, \varphi)$ satisfy $\Diamond\varphi$:

φ-Liveness Test:

Decompose $T_P(s, \varphi)$ into a topologically sorted list of maximal strongly connected components: $K_1, \ldots, K_r$.

For each $i = 1, \ldots, r$ check:

(*a*) If K_i is terminal then it consists of a single node satisfying φ.

(*b*) If K_i is nonterminal, then there must exist a j, $j = 1, \ldots, m$, such that every state $s \in K_i$ has a P_j transition leading out of K_i.

Lemma:

All just computations in $T_P(s, \varphi)$ realize $\Diamond \varphi$ iff the φ-liveness test succeeds.

Proof:

Assume that the test succeeds. Let σ be any maximal computation in $T_P(s, \varphi)$. By the ordering of the $K_1, \ldots, K_r$, from a certain point on, the computation must be fully contained in a single component, K_ℓ say. If K_ℓ is terminal then the computation terminates once it has entered K_ℓ, and the last state satisfies φ by (*a*) above. If K_ℓ is not terminal then being contained in K_ℓ and by (*b*) it must be infinite, since no state in K_ℓ is terminal. Furthermore, no P_j transition is ever taken once the computation has entered K_ℓ, otherwise it would have left K_ℓ. Consequently the computation is unjust with respect to P_j. Thus all just computation must eventually realize φ.

Assume that the test fails. Then either there is a terminal component K_i not satisfying φ, or there exists a nonterminal component K_i not satisfying condition (*b*). In the first case we construct a computation σ leading from σ to K_i, and then either stopping if the state $s \in K_i$ is terminal or looping within K_i in a loop that spans all of K_i. Since states within K_i do not satisfy φ (actually *none of them* does) this can be shown to be a just computation not realizing φ. In the second case, we construct again a computation σ reaching K_i and continuing in a loop spanning all the transitions within K_i. By violation of condition (*b*) every process P_j that has not terminated yet has a P_j transition internal to K_i. Thus by traversing all transitions in K_i, we generate a just computation which does not realize φ.

Note that the construction of T_P, its decomposition into strongly connected components and applying the liveness test are all linear in the size of T_P.

In order to check that $q \supset \Diamond \varphi$ is valid for P we could in principle take each $s \in T_P$ which satisfies q, construct $T_P(s, \varphi)$ and apply the φ-liveness test to it. But we can actually be more efficient as follows:

Let $s_1, \ldots, s_k$ be all the q-states in T_P. Construct $T_P(s_1, \varphi_1)$ and check it for φ_1-liveness, where

$$\varphi_1(s) = \varphi(s).$$

Next, construct $T_P(s_2, \varphi_2)$ and check it for φ_2-liveness, where

$$\varphi_2(s) = \varphi(s) \ \vee \ s \in T_P(s_1, \varphi_1)$$

Thus in constructing $T_P(s_2, \varphi_2)$ we may stop the analysis once the computation enters $T_P(s_1, \varphi_1)$, since we already know that all computations there realize φ.

In general we construct $T_P(s_i, \varphi_i)$ and check it for φ_i-liveness for $i = 1, \ldots, k$ where:

$$\varphi_i(s) = \varphi(s) \ \vee \ [s \in \bigcup_{j<i} T_P(s_j, \varphi_j)].$$

In this way we essentially consider each state at most once and the whole procedure becomes *linear* in $|T_P|$. ∎

Let us apply this procedure for checking validity of $at\,\ell_1 \supset \Diamond\, at\,\ell_3$ on the mutual exclusion program. We will check the following q-states:

$$s_{17} : (\ell_1, m_3, 2), \quad s_{12} : (\ell_1, m_0, 2), \quad s_{13} : (\ell_1, m_1, 2),$$
$$s_1 : (\ell_1, m_0, 1), \quad s_3 : (\ell_1, m_1, 1), \quad s_{16} : (\ell_1, m_2, 2).$$

In Fig. 4 we present $T_P(s_{17}, at\,\ell_3)$. In decomposing the graph we find that every component consists of exactly one node and a possible sorting order for them is:

$$s_{17},\ s_{12},\ s_{13},\ s_{16},\ s_{18},\ s_{19},\ s_4,\ s_5,\ s_6,\ s_8,\ s_9.$$

The terminal components are s_5 and s_9 and they both satisfy $at\,\ell_3$. For every other component we easily identify a helpful process leading out of the component. Thus P_1 is helpful for $\{s_{17},\ s_{12},\ s_{13},\ s_{16},\ s_4,\ s_8\}$ and P_2 is helpful for $\{s_{18},\ s_{19},\ s_6\}$.

Note that this diagram also took care of s_{12}, s_{13}, s_{16}. The next q-state not yet analyzed is s_1. We construct for it $T_P(s_1, \varphi_2)$ where $\varphi_2(s) = at\,\ell_3 \vee s \in T_P(s_{17}, \ell_3)$.

The corresponding diagram in Fig. 5 shows that all computations starting at s_1 or s_3 eventually must enter $T_P(s_{17}, at\,\ell_3)$. Consequently we conclude that $at\,\ell_1 \supset \Diamond\, at\,\ell_3$ is valid for the program P.

12. TESTING UNLESS PROPERTIES

Let the formula to be tested be

$$q \supset (\varphi_r\, \mathcal{U}\, \varphi_{r-1} \ldots \varphi_1\, \mathcal{U}\, \varphi_0).$$

Let $s \in T_P$ be an accessible q-state. Construct $T_P(s, \varphi_0)$ as before. We propose the following test for checking that all computations in $T_P(s, \varphi_0)$ satisfy $w : \varphi_r\, \mathcal{U}\, \varphi_{r-1} \ldots \varphi_1\, \mathcal{U}\, \varphi_0$.

w-Precedence Test:

Decompose $T_P(s, \varphi_0)$ into a topologically sorted list of maximal strongly connected components: $K_1, \ldots, K_r$. Proceeding from K_r down to K_1, we try to assign each component K_i a rank $\rho_i = \rho(K_i)$ as follows:

Let ρ_i be the smallest $k \geq 0$ such that all states in K_i satisfy φ_k and that any component K_j, directly connected to K_i, $i \geq j$, has a lower or equal rank, i.e., $k \geq \rho_j$.

If we fail to rank some component K_i then the test is said to fail, otherwise we say that it has succeeded.

Lemma A:

If the w-precedence test succeeds, then all computations in $T_P(s, \varphi_0)$ satisfy w.

Proof:

Assume that the test succeeded. Let φ be any computation in $T_P(s, \varphi_0)$. Such a computation must progress through a finite chain of components $K_{i_1}, K_{i_2}, \ldots, K_{i_\ell}$, with $i_1 < i_2 < \ldots < i_\ell$. Thus it sucessively satisfies $\varphi_{\rho(K_{i_1})}, \varphi_{\rho(K_{i_2})}, \ldots \varphi_{\rho(K_{i_\ell})}$ with $\rho(K_{i_1}) \geq \rho(K_{i_2}) \geq \ldots \geq \rho(K_{i_\ell})$.

Obviously it satisfies w.

Let K_i be any component. We say that we failed to assign K_i the rank j if either $\rho_i > j$ or we failed to rank K_i altogether. ∎

Lemma B:

If we failed to assign K_i the rank j then for every $s \in K_i$ there exists a computation $\sigma = s \rightarrow \ldots$ (beginning in s) that does not satisfy

$$w_j \;=\; \varphi_j \,\mathfrak{U}\, \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0.$$

Proof:

We will prove the lemma by double induction, first on $j = 0, 1, \ldots$ and then for each j on $i = r, r-1, \ldots, 1$.

Consider first $j = 0$. Let $s \in K_i$ be any state in K_i. If s satisfies φ_0 then K_i consists of s alone and has no successors. Correspondingly we could have defined $\rho(K_i) = 0$. Since we failed to assign 0 to K_i, s does not satisfy φ_0. Consequently any computation beginning in s falsifies $w_0 = \varphi_0$. This establishes the lemma for $j = 0$ and $K_1, \ldots, K_r$.

Consider now a $j > 0$ and assume by induction that the lemma has been proved for $j-1$ and K_i and also for j and each of $K_{i+1}, \ldots, K_r$. Let $s \in K_i$.

There could be two distinct reasons why we failed to assign the rank j to K_i.

- There exists some state $s^1 \in K_i$ which does not satisfy φ_j. By the induction hypothesis there exists a computation $\sigma' = s^1, s^2, \ldots$ which does not satisfy w_{j-1}. We claim that σ' also does not satisfy w_j. For σ' to satisfy w_j there must be a (possibly empty) prefix of σ' continuously satisfying φ_j followed by a suffix which satisfies w_{j-1}. Since s^1 falsifies φ_j, the prefix must be empty and the whole of σ' must satisfy w_{j-1} which contradicts the definition of σ'.

  It only remains to obtain a similar computation starting from s, the arbitrarily specified state in K_i. If by chance $s = s^1$ then σ' will do. Otherwise, since s and s^1 belong to the same strongly connected component there must exist a path $s = s_1, \ldots, s_m = s^1$ within K_i connecting s to s^1. Consider the computation $\sigma = s, \ldots, s^1, s^2, \ldots$, i.e., the path

from s to s^1 followed by σ'. Since no state in K_i satisfies φ_0, σ can satisfy w_j only if σ' does. Thus σ falsifies w_j.

- The second case where we fail to assign j to K_i is that there exists a K_ℓ directly connected to K_i, $i < \ell$, such that $\rho_\ell > j$ or more generally we failed to assign j to K_ℓ. Thus there exists $s_i \in K_i$ and $s_\ell \in K_\ell$ such that

$$s_i \xrightarrow{P_k} s_\ell \quad \text{for some } P_k.$$

By strong connectedness there exists a (possibly empty) path connecting s to s_i : s, $\ldots, s_i$. By the induction hypothesis since $\ell > i$ and we failed to assign j to K_ℓ there exists a computation σ_ℓ : $s_\ell, s^2, \ldots$ which falsifies w_j. Consider now the computation

$$\sigma : \ s, \ \ldots, \ s_i, \ s_\ell, \ s^2, \ \ldots$$

The computation σ consists first of the path from s to s_i within K_i, then the edge from s_i to s_ℓ and then follows σ_ℓ. Since the whole segment $s, \ldots, s_\ell$ does not contain a state satisfying φ_0, σ can satisfy w_j only if σ_ℓ does, which is impossible. Thus σ falsifies w_j as required. ∎

Let now K_i be a component that was not ranked altogether. By the last lemma there exists a computation $\sigma = s, s^2, s^3, \ldots$ with $s \in K_i$ such that σ falsifies

$$w_r \ = \ \varphi_r \,\mathfrak{U} \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0.$$

We can prefix σ by a path leading from s_0 to s and obtain a computation $\sigma_0 = s_0, \ldots, s, \ldots$ which fails to satisfy w_r. We may combine Lemmas A and B to obtain:

Corollary:

Given $T_P(s_0, \varphi_0)$, all s_0-initialized computations in $T_P(s_0, \varphi_0)$ satisfy

$$w = \varphi_r \,\mathfrak{U} \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0$$

iff the w-precedence test succeeded.

Proof:

In order to test the general implication $q \supset w$ on the entire T_P diagram we proceed as follows:

Let $s_1, s_2, \ldots, s_k$ be all the q-states in T_P. Construct $T_P(s_1, \varphi_0)$ and test $\varphi_r \,\mathfrak{U} \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0$ on it. Construct $T_P(s_2, \psi_2)$ where $\psi_2(s) = \varphi_0(s) \vee s \in T_P(s_1, \varphi_0)$.

Test $\varphi_r \,\mathfrak{U} \ldots \varphi_1 \,\mathfrak{U}\, \varphi_0$ on $T_P(s_2, \psi_2)$. In ranking the components we add the following rule:

If K_i is a terminal component consisting of the single node $s \in T_P(s_1, \varphi_0)$, give K_i the rank that s (or the component containing s) has received in $T_P(s_1, \varphi_0)$.

In general we construct $T_P(s_i, \psi_i)$ where

$$\psi_i(s) \ = \ \varphi_0(s) \vee [s \in \bigcup_{j<i} T_P(s_j, \psi_j)] \qquad (\psi_1 = \varphi_0).$$

We then test $\varphi_r \,\mathfrak{U}\, \ldots \,\mathfrak{U}\, \varphi_0$ on $T_P(s_i, \psi_i)$ ranking any component consisting of $s \in T_P(s_j, \psi_j)$ for some $j < i$ according to the rank it received earlier.

Consequently the testing procedure is again linear in the size of T_P. To be precise, of complexity $r \cdot m \cdot |T_P|$. ∎

To illustrate the procedure let us test the validity of the following unless property:

$$\ell_0 \supset (\ell_0 \,\mathfrak{U}\, m_3 \,\mathfrak{U}\, {\sim}m_3 \,\mathfrak{U}\, m_3 \,\mathfrak{U}\, {\sim}m_3 \,\mathfrak{U}\, \ell_3).$$

This property again expresses a certain kind of 2-bounded overtaking. However the reference point is when P_1 is at ℓ_0. It states that from the time P_1 decides to leave ℓ_0, P_2 may enter m_3 at most twice before P_1 enters ℓ_3. Furthermore, actual 2-overtaking can take place only if P_1 on exiting ℓ_0 finds P_2 in m_3 at precisely the same moment. If on exiting ℓ_0, P_1 find P_2 anywhere else then at most 1-overtaking can take place. In contrast with other *unless* properties considered before in this paper, this property is not an *until* property. The corresponding until property does not hold since when P_1 is at ℓ_0 it is quite acceptable that it never gets out to achieve ℓ_3.

We define

$$q = \varphi_5 : \ at\,\ell_0$$

$$\varphi_4 = \varphi_2 : \ at\,m_3$$

$$\varphi_3 = \varphi_1 : \ {\sim}at\,m_3$$

$$\varphi_0 = at\,\ell_3$$

Accessible q-states in T_P are:

$$s_{15} : (\ell_0, m_3, 2), \quad s_{10} : (\ell_0, m_0, 2), \quad s_{11} : (\ell_0, m_1, 2),$$
$$s_{14} : (\ell_0, m_2, 2), \quad s_0 : (\ell_0, m_0, 1), \quad s_2 : (\ell_0, m_1, 1).$$

In Fig. 6 we have $T_P(s_{15}, \varphi_0)$. Its component decomposition gives the following topologically sorted list of components:

$$K_1 \ = \ \{s_{15}, s_{10}, s_{11}, s_{14}\}, \{s_{17}\}, \{s_{12}\}, \{s_{13}\}, \{s_{16}\}, \{s_{18}\}, \{s_{19}\}, \{s_4\}, \{s_5\}, \{s_6\}, \{s_8\}, \{s_9\}.$$

Going backwards we assign the following ranks:

$$\rho_i = 0 \quad \text{for } i \in \{5, 9\}$$

$$\rho_i = 1 \quad \text{for } i \in \{8, 6, 4\}$$

$$\rho_i = 2 \quad \text{for } i = 19$$

$$\rho_i = 3 \quad \text{for } i \in \{18, 16, 13, 12\}$$

$$\rho_i = 4 \quad \text{for } i = 17$$

$$\rho(K_1) = 5$$

This shows that the desired unless property actually holds for the q-states $s_{15}, s_{10}, s_{11}, s_{14}$.

Next let us consider $T_P(s_0,\ [\varphi_0(s) \vee s \in T_P(s_{15}, \varphi_0)])$. It is given in Fig. 7. All the terminal nodes belong to the previous diagram and their ranks have been listed. We may proceed to rank the unranked states in $T_P(s_0, \psi_2)$.

We define

$$\rho_i = 3 \quad \text{for } i \in \{1,3\},$$

and

$$\rho_i = 5 \quad \text{for } i \in \{0,2\}.$$

Thus, all q-states have been successfully ranked, and the unless property:

$$\ell_0 \supset (\ell_0 \,\mathfrak{U}\, m_3 \,\mathfrak{U}\, {\sim}m_3 \,\mathfrak{U}\, m_3 \,\mathfrak{U}\, {\sim}m_3 \,\mathfrak{U}\, \ell_3).$$

has been established. We obviously cannot do better since the computation:

$$s_{15} \rightarrow s_{17} \rightarrow s_{12} \rightarrow s_{13} \rightarrow s_{16} \rightarrow s_{18} \rightarrow s_{19} \rightarrow s_4 \rightarrow s_5$$

demonstrates 2-overtaking.

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13. REFERENCES

[CES] Clarke, E.M., E.A. Emerson, and A.P. Sistla, "Automatic Verification of Finite State Concurrent Systems using Temporal Logic Specifications: A Practical Approach," Proc. of the IEEE Conf. on Foundations of Computer Science, Chicago (1982).

[K] Keller, R.M., "Formal verification of parallel programs," CACM, Vol. 19, No. 7 (July 1976), pp. 371–384.

[L1] Lamport, L., "Proving the Correctness of Multiprocess Programs," IEEE Trans. Soft. Eng. SE-3, 2 (Mar. 1977), pp. 125-143.

[L2] Lamport, L., " 'Sometime' is Sometimes 'Not Never': On the Temporal Logic of Programs," 7th Annual ACM Symposium on Principles of Programming Languages (1980), pp. 174-185.

[LPS] Lehmann, D., A. Pnueli, and J. Stavi, "Impartiality, justice and fairness: the ethics of concurrent termination," in *Automata Languages and Programming*, Lecture Notes in Computer Science 115, Springer Verlag (1981), pp. 264–277.

[MP1] Manna, Z. and A. Pnueli, "Verification of Concurrent Prorams: The Temporal Framework," in *The Correctness Problem in Computer Science* (R.S. Boyer and J S. Moore, eds.), International Lecture Series in Computer Science, Academic Press, London (1982), pp. 215-273.

[MP2] Manna, Z. and A. Pnueli, "Verification of Concurrent Programs: Temporal Proof Principles," Proc. of the Workshop on Logic of Programs (D. Kozen, ed.), Yorktown-Heights, N.Y. (1981). Springer-Verlag Lecture Notes in Computer Science 131, pp. 200-252.

[MP3] Manna, Z. and A. Pnueli, "Verification of Concurrent Programs: Proving Eventualities by Well-Founded Ranking," TOPLAS (1983, to appear).

[MP4] Manna, Z. and A. Pnueli, "Verification of Concurrent Programs: a Temporal Proof System," Proc. 4th School on Advanced Programming, Amsterdam, Holland (June 1982).

[MP5] Manna, Z. and A. Pnueli, "How to Cook a Temporal Proof System for Your Pet Language," in the Proc. of the Symposium on Principles of Programming Languages, Austin, Texas (Jan. 1983).

[OL] Owicki, S. and L. Lamport, "Proving Liveness Properties of Concurrent Programs," ACM Transactions on Programming Languages and Systems, Vol. 4, No. 3 (July 1982), pp. 455-495.

[OG] Owicki, S. and D. Gries, "An Axiomatic Proof technique for Parallel Programs," Acta Informatica, Vol. 6, No. 4 (1976), pp. 319-340.

[Pe] Peterson, G.L., "Myths about the Mutual Exclusion Problem," Information Processing Letters, Vol. 12, No. 3 (June 1981), pp. 115-116.

[PS] Pnueli, and A., Sherman R., "Semantic Tableau for Temporal Logic," Technical Report, CS81-21, The Weizmann Institute (Sept. 81).

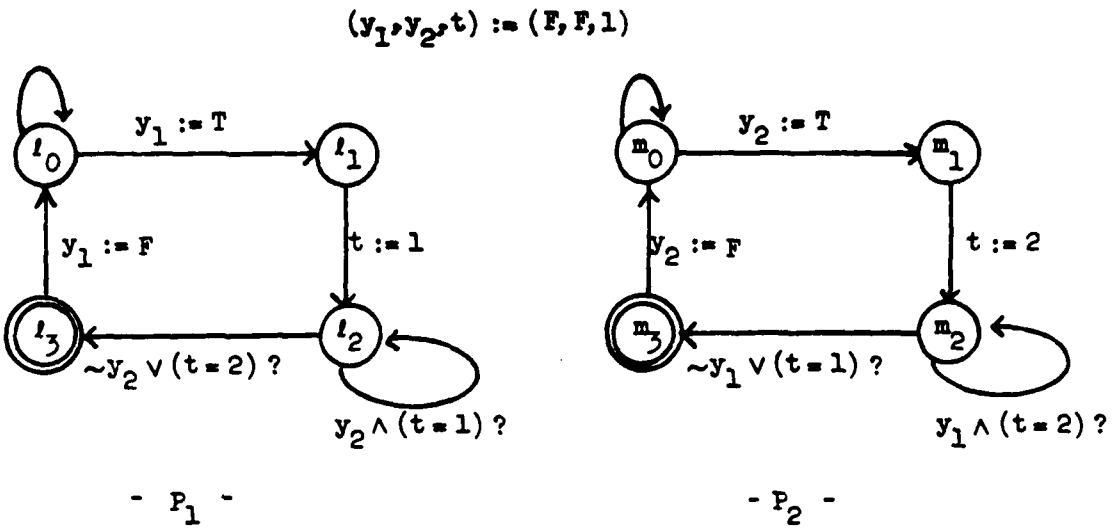

Figure 0

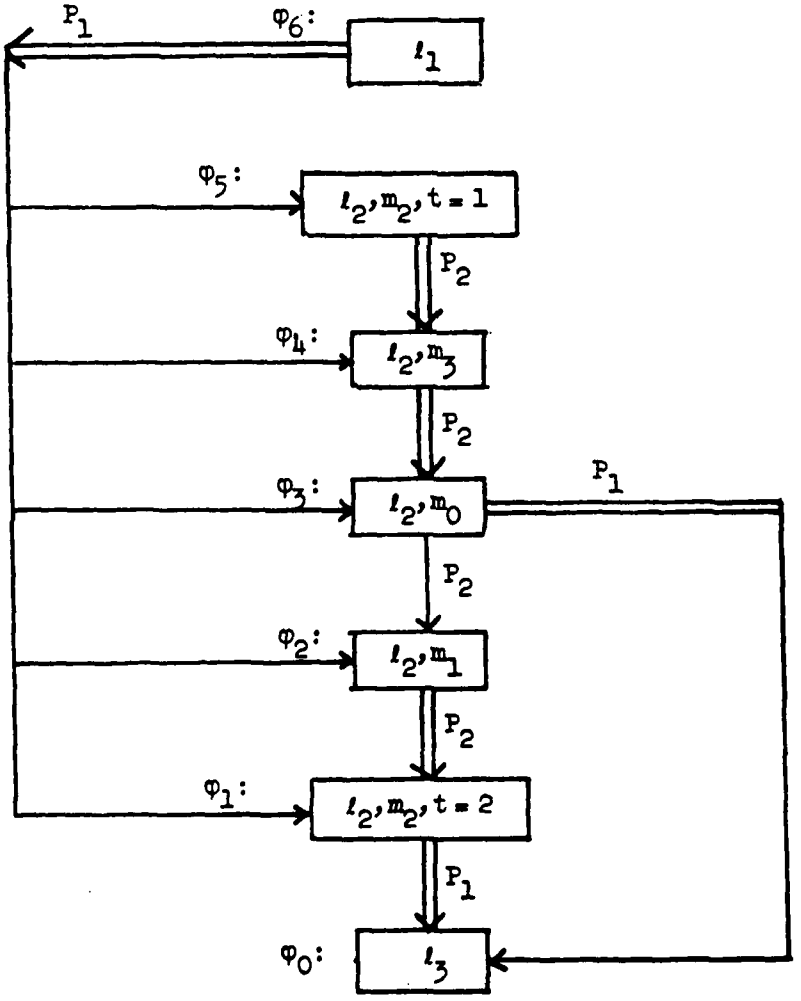

Fig. 1. Proof Diagram for $\vdash \ell_1 \supset \Diamond \ell_3$

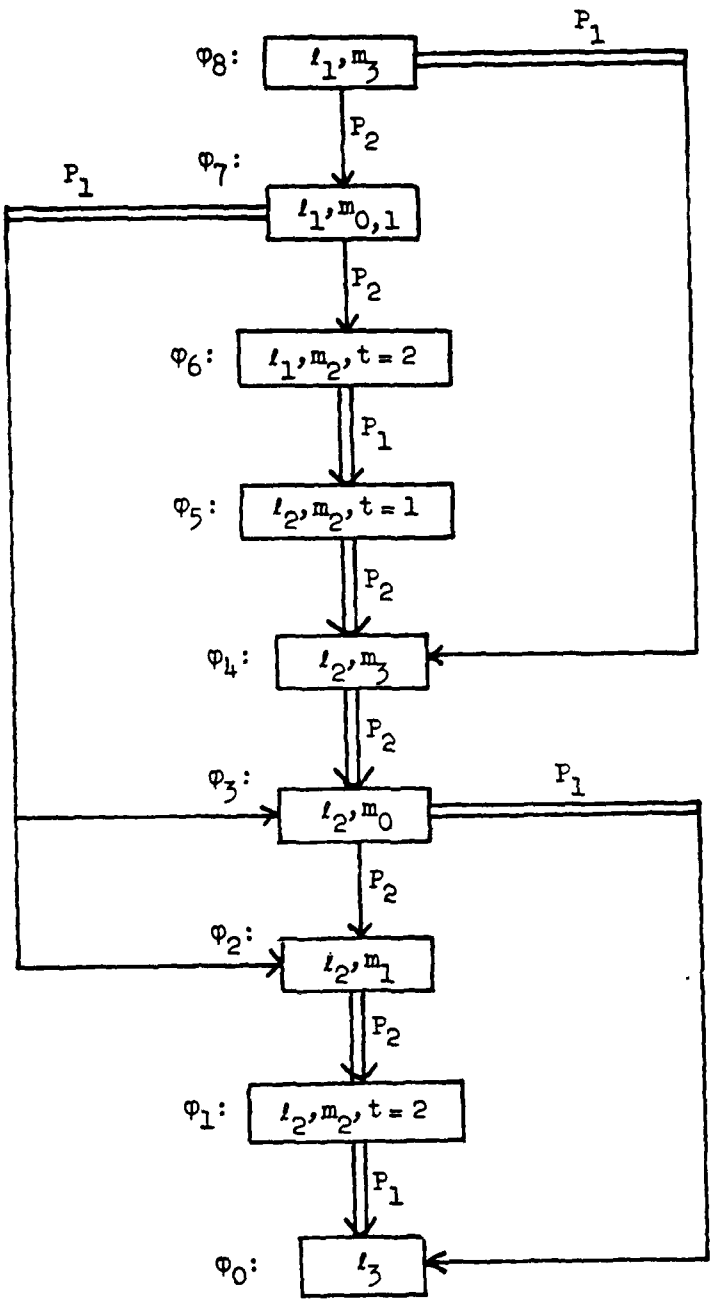

Fig. 2. Proof Diagram for 2-bounded overtaking from ℓ_1

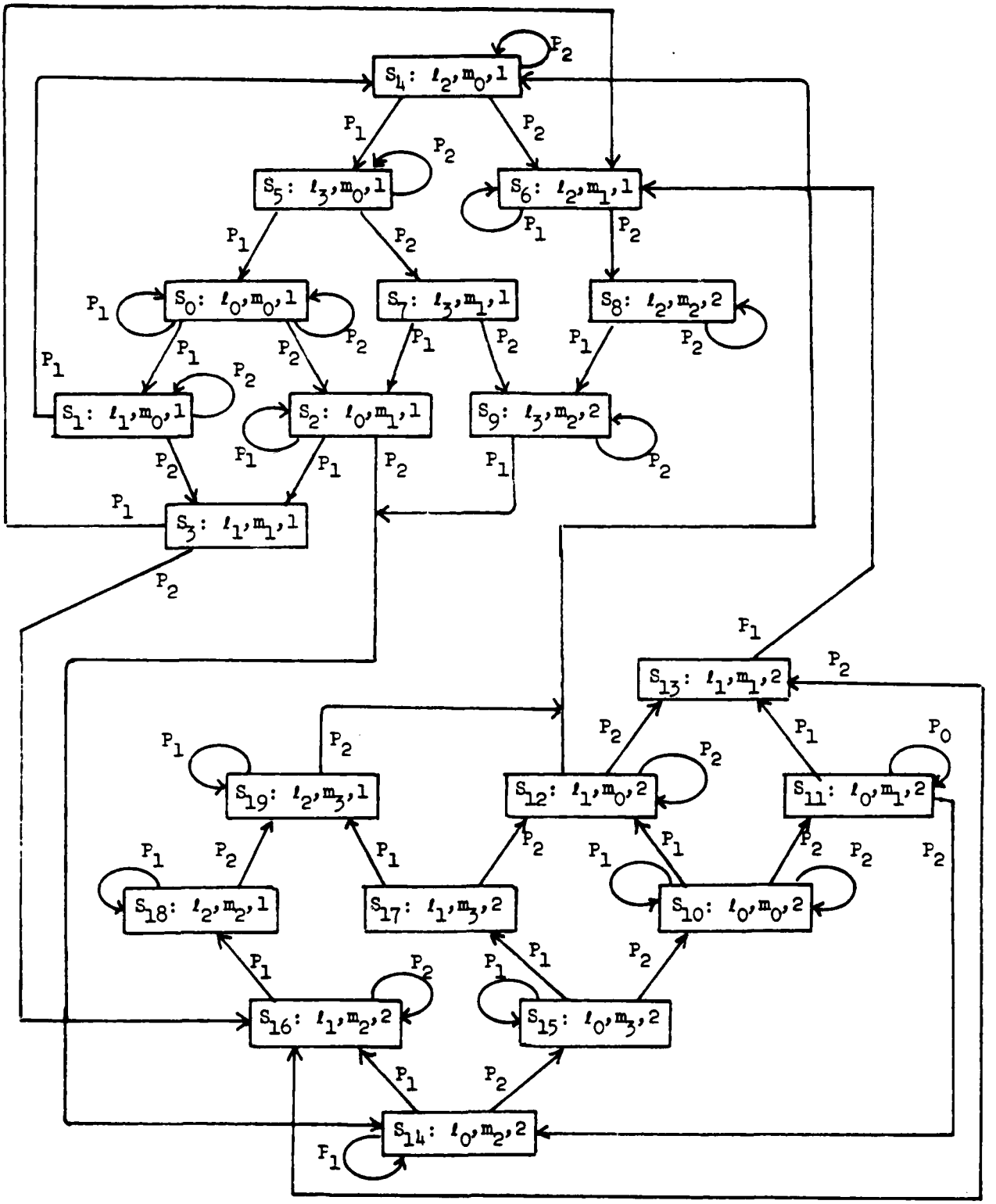

Fig. 3. Joint Transition Diagram for the Mutual Exclusion Program.

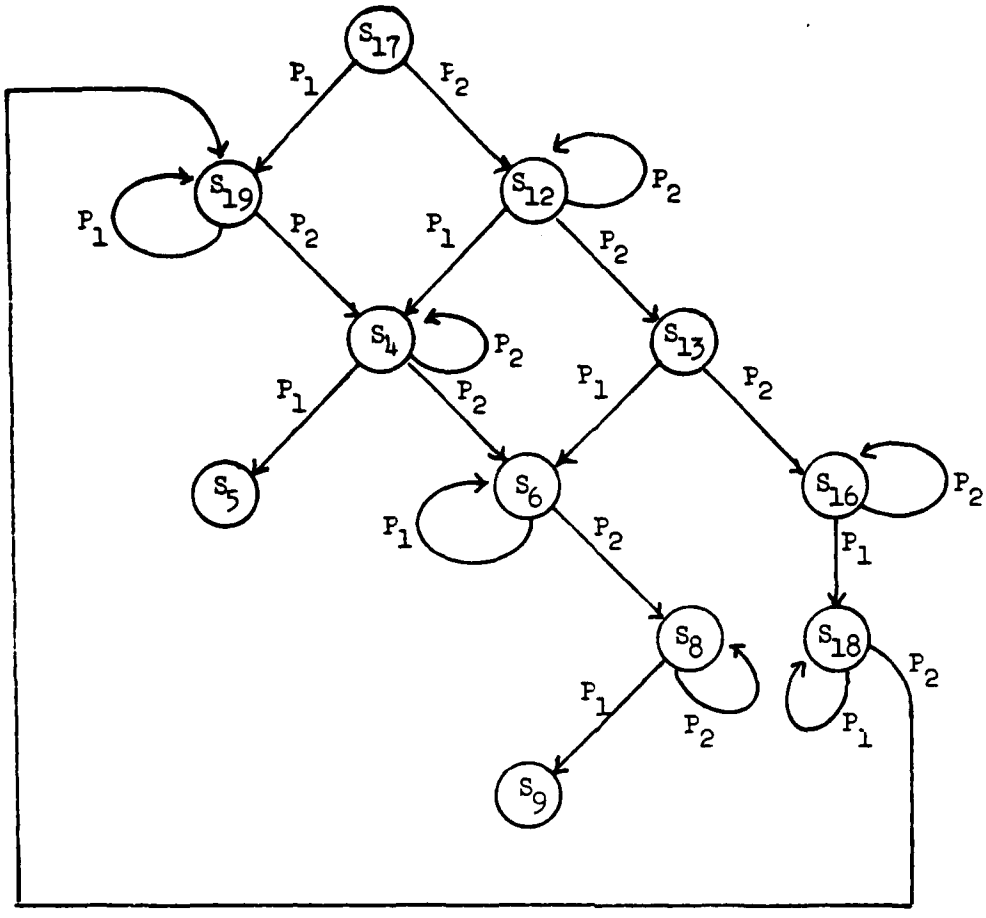

Fig. 4. $T_P(S_{17}, \underline{at}\ \ell_3)$

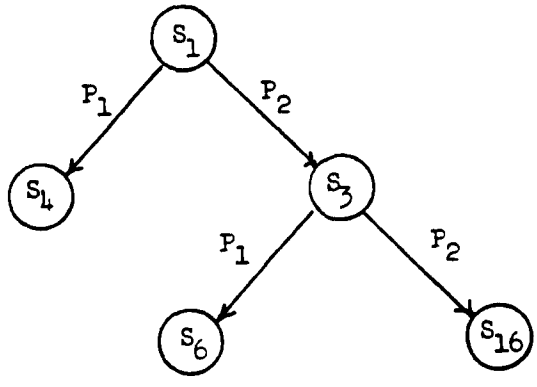

Fig. 5. $T_P(S_1, \varphi_2)$

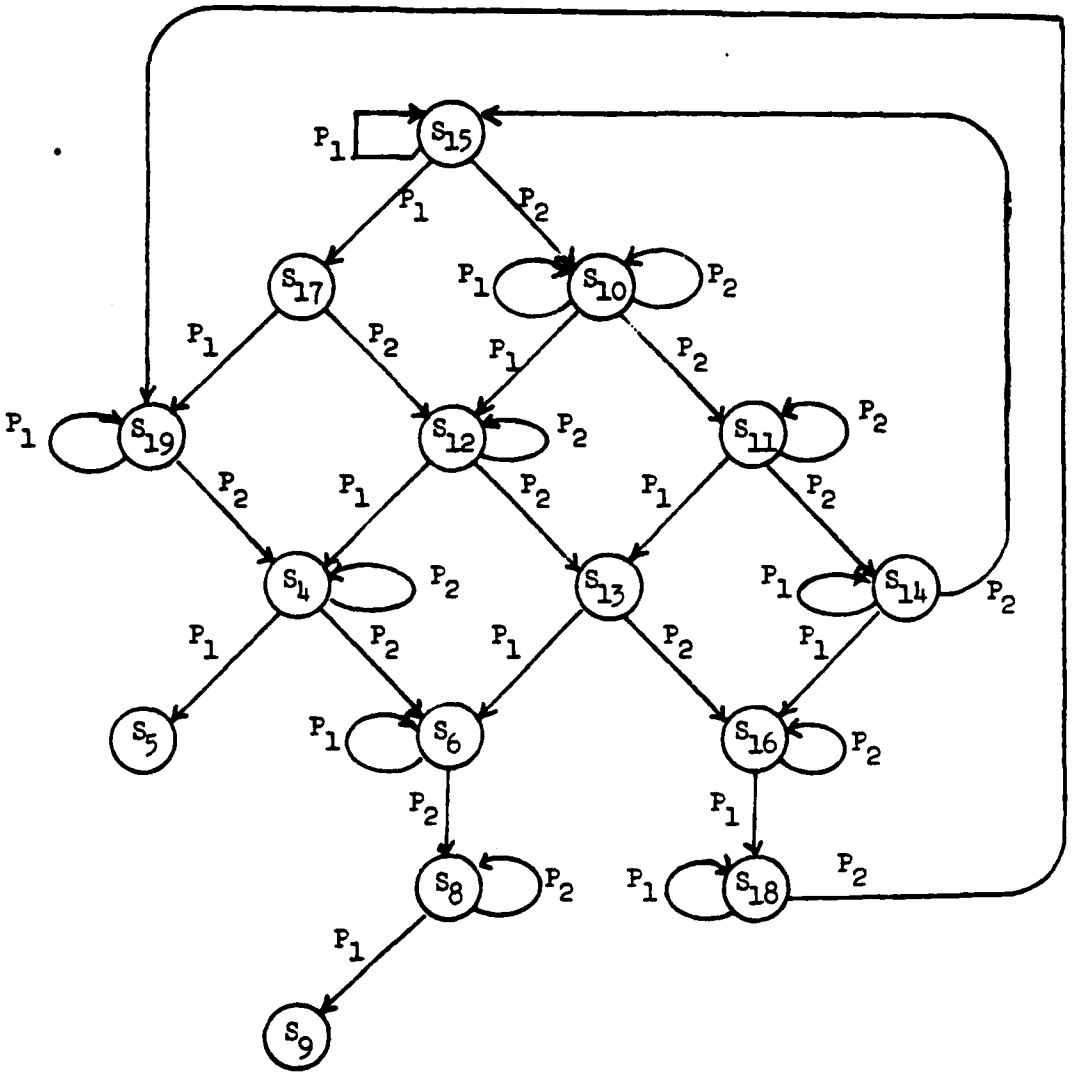

Fig. 6. $T_P(S_{15}, \Phi_0)$

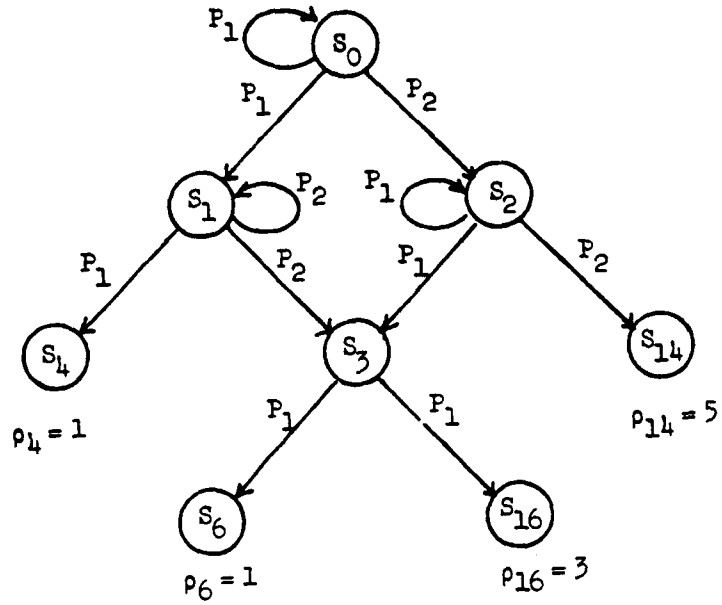

Fig. 7. $T_P(S_0, \Psi_2)$


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